EXAMPLE 1 Evaluating Trigonometric Functions

Let $P = (-3, -5)$ be a point on the terminal side of θ. Find each of the six trigonometric functions of θ.

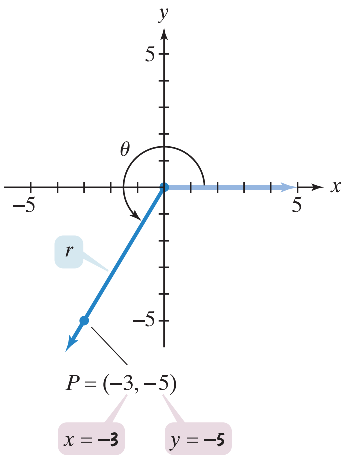

Figure 4.42

Solution The situation is shown in **Figure 4.42**. We need values for x, y, and r to evaluate all six trigonometric functions. We are given the values of x and y. Because $P = (-3, -5)$ is a point on the terminal side of θ, $x = -3$ and $y = -5$. Furthermore,

$$r = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + (-5)^2} = \sqrt{9 + 25} = \sqrt{34}.$$

Now that we know x, y, and r, we can find the six trigonometric functions of θ. Where appropriate, we will rationalize denominators.

$$\sin\theta = \frac{y}{r} = \frac{-5}{\sqrt{34}} = -\frac{5}{\sqrt{34}} \cdot \frac{\sqrt{34}}{\sqrt{34}} = -\frac{5\sqrt{34}}{34} \qquad \csc\theta = \frac{r}{y} = \frac{\sqrt{34}}{-5} = -\frac{\sqrt{34}}{5}$$

$$\cos\theta = \frac{x}{r} = \frac{-3}{\sqrt{34}} = -\frac{3}{\sqrt{34}} \cdot \frac{\sqrt{34}}{\sqrt{34}} = -\frac{3\sqrt{34}}{34} \qquad \sec\theta = \frac{r}{x} = \frac{\sqrt{34}}{-3} = -\frac{\sqrt{34}}{3}$$

$$\tan\theta = \frac{y}{x} = \frac{-5}{-3} = \frac{5}{3} \qquad \cot\theta = \frac{x}{y} = \frac{-3}{-5} = \frac{3}{5}$$

Check Point 1 Let $P = (1, -3)$ be a point on the terminal side of θ. Find each of the six trigonometric functions of θ.

How do we find the values of the trigonometric functions for a quadrantal angle? First, draw the angle in standard position. Second, choose a point P on the angle's terminal side. The trigonometric function values of θ depend only on the size of θ and not on the distance of point P from the origin. Thus, we will choose a point that is 1 unit from the origin. Finally, apply the definitions of the appropriate trigonometric functions.

EXAMPLE 2 Trigonometric Functions of Quadrantal Angles

Evaluate, if possible, the sine function and the tangent function at the following four quadrantal angles:

a. $\theta = 0° = 0$ **b.** $\theta = 90° = \frac{\pi}{2}$ **c.** $\theta = 180° = \pi$ **d.** $\theta = 270° = \frac{3\pi}{2}$.

Solution

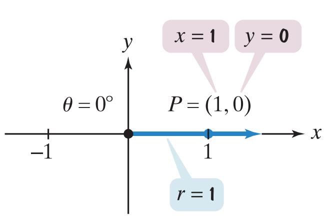

Figure 4.43

a. If $\theta = 0° = 0$ radians, then the terminal side of the angle is on the positive x-axis. Let us select the point $P = (1, 0)$ with $x = 1$ and $y = 0$. This point is 1 unit from the origin, so $r = 1$. **Figure 4.43** shows values of x, y, and r corresponding to $\theta = 0°$ or 0 radians. Now that we know x, y, and r, we can apply the definitions of the sine and tangent functions.

$$\sin 0° = \sin 0 = \frac{y}{r} = \frac{0}{1} = 0$$

$$\tan 0° = \tan 0 = \frac{y}{x} = \frac{0}{1} = 0$$

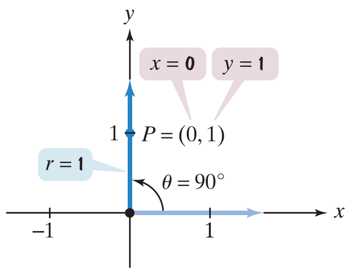

Figure 4.44

b. If $\theta = 90° = \frac{\pi}{2}$ radians, then the terminal side of the angle is on the positive y-axis. Let us select the point $P = (0, 1)$ with $x = 0$ and $y = 1$. This point is 1 unit from the origin, so $r = 1$. **Figure 4.44** shows values of x, y, and r corresponding to $\theta = 90°$ or $\frac{\pi}{2}$. Now that we know x, y, and r, we can apply the definitions of the sine and tangent functions.

$$\sin 90° = \sin\frac{\pi}{2} = \frac{y}{r} = \frac{1}{1} = 1$$

$$\tan 90° = \tan\frac{\pi}{2} = \frac{y}{x} = \frac{1}{0}$$

Because division by 0 is undefined, tan 90° is undefined.

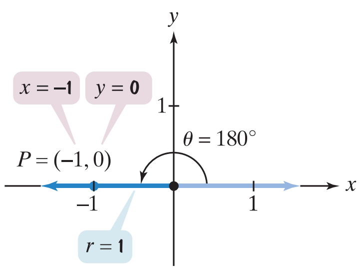

Figure 4.45

c. If $\theta = 180° = \pi$ radians, then the terminal side of the angle is on the negative x-axis. Let us select the point $P = (-1, 0)$ with $x = -1$ and $y = 0$. This point is 1 unit from the origin, so $r = 1$. **Figure 4.45** shows values of x, y, and r corresponding to $\theta = 180°$ or π. Now that we know x, y, and r, we can apply the definitions of the sine and tangent functions.

$$\sin 180° = \sin\pi = \frac{y}{r} = \frac{0}{1} = 0$$

$$\tan 180° = \tan\pi = \frac{y}{x} = \frac{0}{-1} = 0$$

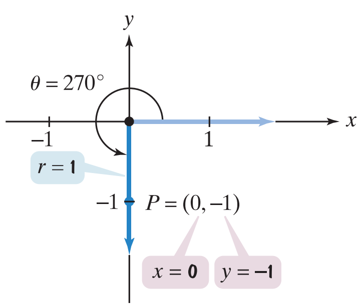

Figure 4.46

d. If $\theta = 270° = \frac{3\pi}{2}$ radians, then the terminal side of the angle is on the negative y-axis. Let us select the point $P = (0, -1)$ with $x = 0$ and $y = -1$. This point is 1 unit from the origin, so $r = 1$. **Figure 4.46** shows values of x, y, and r corresponding to $\theta = 270°$ or $\frac{3\pi}{2}$. Now that we know x, y, and r, we can apply the definitions of the sine and tangent functions.

$$\sin 270° = \sin\frac{3\pi}{2} = \frac{y}{r} = \frac{-1}{1} = -1$$

$$\tan 270° = \tan\frac{3\pi}{2} = \frac{y}{x} = \frac{-1}{0}$$

Because division by 0 is undefined, tan 270° is undefined.

Discovery

Try finding tan 90° and tan 270° with your calculator. Describe what occurs.

Check Point 2 Evaluate, if possible, the cosine function and the cosecant function at the following four quadrantal angles:

a. $\theta = 0° = 0$ **b.** $\theta = 90° = \frac{\pi}{2}$ **c.** $\theta = 180° = \pi$ **d.** $\theta = 270° = \frac{3\pi}{2}$.

2 Use the signs of the trigonometric functions.

The Signs of the Trigonometric Functions

In Example 2, we evaluated trigonometric functions of quadrantal angles. However, we will now return to the trigonometric functions of nonquadrantal angles. **If θ is not a quadrantal angle, the sign of a trigonometric function depends on the quadrant in which θ lies.** In all four quadrants, r is positive. However, x and y can be positive or negative. For example, if θ lies in quadrant II, x is negative and y is positive. Thus, the only positive ratios in this quadrant are $\frac{y}{r}$ and its reciprocal, $\frac{r}{y}$. These ratios are the function values for the sine and cosecant, respectively. In short, if θ lies in quadrant II, $\sin\theta$ and $\csc\theta$ are positive. The other four trigonometric functions are negative.

Figure 4.47 summarizes the signs of the trigonometric functions. If θ lies in quadrant I, all six functions are positive. If θ lies in quadrant II, only $\sin\theta$ and $\csc\theta$ are positive. If θ lies in quadrant III, only $\tan\theta$ and $\cot\theta$ are positive. Finally, if θ lies in quadrant IV, only $\cos\theta$ and $\sec\theta$ are positive. Observe that the positive functions in each quadrant occur in reciprocal pairs.

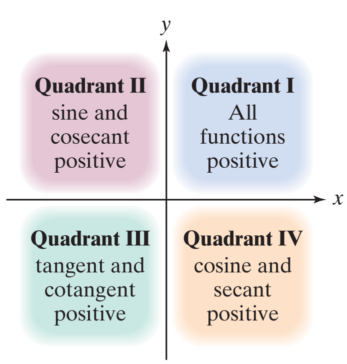

Figure 4.47 The signs of the trigonometric functions

Study Tip

Here's a phrase to help you remember the signs of the trig functions:

A	Smart	Trig	Class.
All trig functions are positive in **QI**.	Sine and its reciprocal, cosecant, are positive in **QII**.	Tangent and its reciprocal, cotangent, are positive in **QIII**.	Cosine and its reciprocal, secant, are positive in **QIV**.

EXAMPLE 3 Finding the Quadrant in Which an Angle Lies

If $\tan\theta < 0$ and $\cos\theta > 0$, name the quadrant in which angle θ lies.

Solution When $\tan\theta < 0$, θ lies in quadrant II or IV. When $\cos\theta > 0$, θ lies in quadrant I or IV. When both conditions are met ($\tan\theta < 0$ and $\cos\theta > 0$), θ must lie in quadrant IV.

Check Point 3 If $\sin\theta < 0$ and $\cos\theta < 0$, name the quadrant in which angle θ lies.

EXAMPLE 4 Evaluating Trigonometric Functions

Given $\tan\theta = -\frac{2}{3}$ and $\cos\theta > 0$, find $\cos\theta$ and $\csc\theta$.

Solution Because the tangent is negative and the cosine is positive, θ lies in quadrant IV. This will help us to determine whether the negative sign in $\tan\theta = -\frac{2}{3}$ should be associated with the numerator or the denominator. Keep in mind that in quadrant IV, x is positive and y is negative. Thus,

In quadrant IV, y is negative.

$$\tan\theta = -\frac{2}{3} = \frac{y}{x} = \frac{-2}{3}.$$

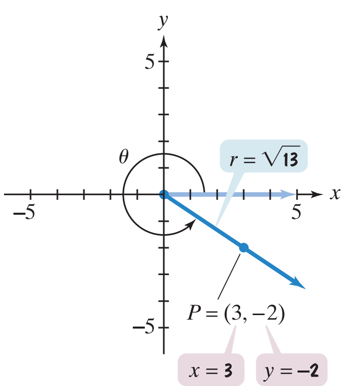

Figure 4.48 $\tan\theta = -\frac{2}{3}$ and $\cos\theta > 0$

(See **Figure 4.48**.) Thus, $x = 3$ and $y = -2$. Furthermore,

$$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + (-2)^2} = \sqrt{9 + 4} = \sqrt{13}.$$

Now that we know x, y, and r, we can find $\cos\theta$ and $\csc\theta$.

$$\cos\theta = \frac{x}{r} = \frac{3}{\sqrt{13}} = \frac{3}{\sqrt{13}} \cdot \frac{\sqrt{13}}{\sqrt{13}} = \frac{3\sqrt{13}}{13} \qquad \csc\theta = \frac{r}{y} = \frac{\sqrt{13}}{-2} = -\frac{\sqrt{13}}{2}$$

Check Point 4 Given $\tan\theta = -\frac{1}{3}$ and $\cos\theta < 0$, find $\sin\theta$ and $\sec\theta$.

In Example 4, we used the quadrant in which θ lies to determine whether a negative sign should be associated with the numerator or the denominator. Here's a situation, similar to Example 4, where negative signs should be associated with *both* the numerator and the denominator:

$$\tan\theta = \frac{3}{5} \quad \text{and} \quad \cos\theta < 0.$$

Because the tangent is positive and the cosine is negative, θ lies in quadrant III. In quadrant III, x is negative and y is negative. Thus,

$$\tan\theta = \frac{3}{5} = \frac{y}{x} = \frac{-3}{-5}.$$

We see that $x = -5$ and $y = -3$.