Discovery

Draw the two right triangles involving 30°, 45°, and 60°. Indicate the length of each side. Use these lengths to verify the function values for the reference angles in the solution to Example 7.

EXAMPLE 7 Using Reference Angles to Evaluate Trigonometric Functions

Use reference angles to find the exact value of each of the following trigonometric functions:

a. $\sin 135°$ **b.** $\cos \frac{4\pi}{3}$ **c.** $\cot\left(-\frac{\pi}{3}\right)$.

Solution

a. We use our two-step procedure to find $\sin 135°$.

Step 1 Find the reference angle, θ', and $\sin \theta'$. **Figure 4.57** shows 135° lies in quadrant II. The reference angle is

$$\theta' = 180° - 135° = 45°.$$

The function value for the reference angle is $\sin 45° = \frac{\sqrt{2}}{2}$.

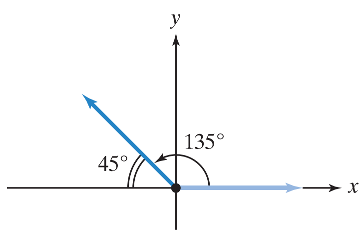

Figure 4.57 Reference angle for 135°

Step 2 Use the quadrant in which θ lies to prefix the appropriate sign to the function value in step 1. The angle $\theta = 135°$ lies in quadrant II. Because the sine is positive in quadrant II, we put a + sign before the function value of the reference angle. Thus,

The sine is positive in quadrant II.

$$\sin 135° = +\sin 45° = \frac{\sqrt{2}}{2}.$$

The reference angle for 135° is 45°.

b. We use our two-step procedure to find $\cos \frac{4\pi}{3}$.

Step 1 Find the reference angle, θ', and $\cos \theta'$. **Figure 4.58** shows that $\theta = \frac{4\pi}{3}$ lies in quadrant III. The reference angle is

$$\theta' = \frac{4\pi}{3} - \pi = \frac{4\pi}{3} - \frac{3\pi}{3} = \frac{\pi}{3}.$$

The function value for the reference angle is

$$\cos \frac{\pi}{3} = \frac{1}{2}.$$

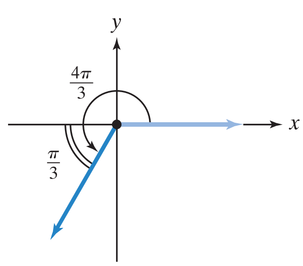

Figure 4.58 Reference angle for $\frac{4\pi}{3}$

Step 2 Use the quadrant in which θ lies to prefix the appropriate sign to the function value in step 1. The angle $\theta = \frac{4\pi}{3}$ lies in quadrant III. Because only the tangent and cotangent are positive in quadrant III, the cosine is negative in this quadrant. We put a − sign before the function value of the reference angle. Thus,

The cosine is negative in quadrant III.

$$\cos \frac{4\pi}{3} = -\cos \frac{\pi}{3} = -\frac{1}{2}.$$

The reference angle for $\frac{4\pi}{3}$ is $\frac{\pi}{3}$.

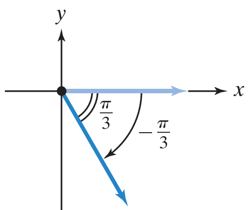

Figure 4.59 Reference angle for $-\frac{\pi}{3}$

c. We use our two-step procedure to find $\cot\left(-\frac{\pi}{3}\right)$.

Step 1 Find the reference angle, θ', and $\cot\theta'$. Figure 4.59 shows that $\theta = -\frac{\pi}{3}$ lies in quadrant IV. The reference angle is $\theta' = \frac{\pi}{3}$. The function value for the reference angle is $\cot\frac{\pi}{3} = \frac{\sqrt{3}}{3}$.

Step 2 Use the quadrant in which θ lies to prefix the appropriate sign to the function value in step 1. The angle $\theta = -\frac{\pi}{3}$ lies in quadrant IV. Because only the cosine and secant are positive in quadrant IV, the cotangent is negative in this quadrant. We put a − sign before the function value of the reference angle. Thus,

The cotangent is negative in quadrant IV.

$$\cot\left(-\frac{\pi}{3}\right) = -\cot\frac{\pi}{3} = -\frac{\sqrt{3}}{3}.$$

The reference angle for $-\frac{\pi}{3}$ is $\frac{\pi}{3}$.

Check Point 7 Use reference angles to find the exact value of the following trigonometric functions:

a. $\sin 300°$ **b.** $\tan\frac{5\pi}{4}$ **c.** $\sec\left(-\frac{\pi}{6}\right)$.

In our final example, we use positive coterminal angles less than 2π to find the reference angles.

EXAMPLE 8 Using Reference Angles to Evaluate Trigonometric Functions

Use reference angles to find the exact value of each of the following trigonometric functions:

a. $\tan\frac{14\pi}{3}$ **b.** $\sec\left(-\frac{17\pi}{4}\right)$.

Solution

a. We use our two-step procedure to find $\tan\frac{14\pi}{3}$.

Step 1 Find the reference angle, θ', and $\tan\theta'$. Because the given angle, $\frac{14\pi}{3}$ or $4\frac{2}{3}\pi$, exceeds 2π, subtract 4π to find a positive coterminal angle less than 2π.

$$\theta = \frac{14\pi}{3} - 4\pi = \frac{14\pi}{3} - \frac{12\pi}{3} = \frac{2\pi}{3}$$

Figure 4.60 shows $\theta = \frac{2\pi}{3}$ in standard position. The angle lies in quadrant II. The reference angle is

$$\theta' = \pi - \frac{2\pi}{3} = \frac{3\pi}{3} - \frac{2\pi}{3} = \frac{\pi}{3}.$$

The function value for the reference angle is $\tan\frac{\pi}{3} = \sqrt{3}$.

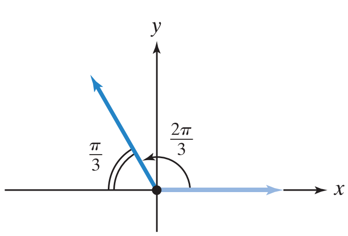

Figure 4.60 Reference angle for $\frac{2\pi}{3}$

Step 2 Use the quadrant in which θ lies to prefix the appropriate sign to the function value in step 1. The coterminal angle $\theta = \frac{2\pi}{3}$ lies in quadrant II. Because the tangent is negative in quadrant II, we put a $-$ sign before the function value of the reference angle. Thus,

The tangent is negative in quadrant II.

$$\tan\frac{14\pi}{3} = \tan\frac{2\pi}{3} = -\tan\frac{\pi}{3} = -\sqrt{3}.$$

The reference angle for $\frac{2\pi}{3}$ is $\frac{\pi}{3}$.

b. We use our two-step procedure to find $\sec\left(-\frac{17\pi}{4}\right)$.

Step 1 Find the reference angle, θ', and $\sec\theta'$. Because the given angle, $-\frac{17\pi}{4}$ or $-4\frac{1}{4}\pi$, is less than -2π, add 6π (three multiples of 2π) to find a positive coterminal angle less than 2π.

$$\theta = -\frac{17\pi}{4} + 6\pi = -\frac{17\pi}{4} + \frac{24\pi}{4} = \frac{7\pi}{4}$$

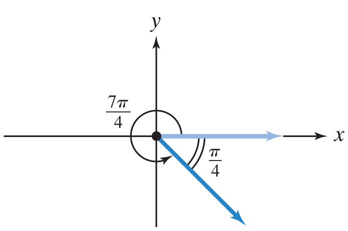

Figure 4.61 Reference angle for $\frac{7\pi}{4}$

Figure 4.61 shows $\theta = \frac{7\pi}{4}$ in standard position. The angle lies in quadrant IV. The reference angle is

$$\theta' = 2\pi - \frac{7\pi}{4} = \frac{8\pi}{4} - \frac{7\pi}{4} = \frac{\pi}{4}.$$

The function value for the reference angle is $\sec\frac{\pi}{4} = \sqrt{2}$.

Step 2 Use the quadrant in which θ lies to prefix the appropriate sign to the function value in step 1. The coterminal angle $\theta = \frac{7\pi}{4}$ lies in quadrant IV. Because the secant is positive in quadrant IV, we put a $+$ sign before the function value of the reference angle. Thus,

The secant is positive in quadrant IV.

$$\sec\left(-\frac{17\pi}{4}\right) = \sec\frac{7\pi}{4} = +\sec\frac{\pi}{4} = \sqrt{2}.$$

The reference angle for $\frac{7\pi}{4}$ is $\frac{\pi}{4}$.

Check Point 8 Use reference angles to find the exact value of each of the following trigonometric functions:

a. $\cos\frac{17\pi}{6}$ **b.** $\sin\left(-\frac{22\pi}{3}\right)$.