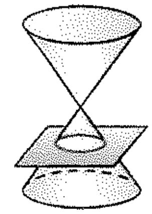
Chapter 14: Conic Sections

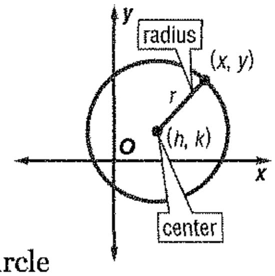
A CONIC SECTION IS A CURVE YOU GET BY INTERSECTING A PLANE & A DOUBLE CONE.



14.2 ~ Circles

Objectives:

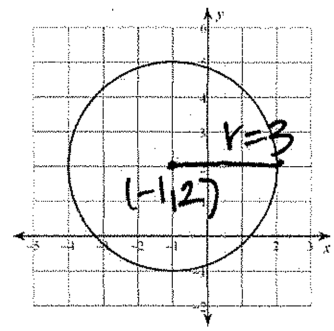
- Write the standard form equation of a circle given points on the circle or its graph
- Given the equation of a circle in general form, complete the square to find the center & radius



- Standard Form of the Equation of a Circle
  - $(x-h)^2+(y-k)^2=r^2$
    - Center: (h, k)
    - Radius: r

- Finding the Standard Form Equation of a Circle
  - Given: the graph of a circle
    - Consider the circle shown.
      - Identify the center of the circle: $(-1,2)$
      - Identify the radius of the circle: $r=3$
    - Write the standard form equation of the circle:

$$(x+1)^2+(y-2)^2=9$$



  - Given: the center (h, k) & a point on the circle (x, y)
    - Consider a circle whose center is $(2,-5)$ and that passes through the point $(-7,-1)$.
      - Find the radius of the circle. x_1 y_1 x_2 y_2

        The radius of a circle is a line segment with one endpoint on the circle and one at the center.

        Use the distance formula: $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$$r=\sqrt{(\underbrace{-7-2}_{-9})^2+(\underbrace{-1+5}_{4})^2}=\sqrt{81+16}=\sqrt{97}$$

      - Write the standard form equation of the circle: $(x-2)^2+(y+5)^2=97$

  - Given: the endpoints of the diameter

    The diameter of a circle is a line segment with each endpoint on the circle that passes through the center of the circle.
    - Consider a circle with a diameter whose endpoints are $(18,-13)$ & $(4,-3)$.
      - Find the center of the circle.

        Use the midpoint formula: $(x_m, y_m)=\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)=\left(\frac{\overbrace{18+4}^{22}}{2}, \frac{\overbrace{-13+(-3)}^{-16}}{2}\right)=(11,-8)$

      - Find the radius of the circle.

$$r=\sqrt{(\underbrace{18-11}_{7})^2+(\underbrace{-13+8}_{-5})^2}=\sqrt{49+25}=\sqrt{74}$$

$$(x-11)^2+(y+8)^2=74$$

- Write the standard form equation of the circle:

❖ General Form Equation of a Circle

- ➢ $Ax^2 + By^2 + Cx + Dy + E = 0$, where A, B, C, and D are constants, $A = B$, and $x \neq y$.
  - In order to identify the center and radius of a circle written in general form, it is necessary to rewrite the equation in standard form.

❖ General Form → Standard Form

- ➢ Completing the Square
  - Move the constant term to the right-hand side.
  - Sort/organize the left-hand side, leaving blanks after each linear term.
  - Set up your squares.
  - Talk half the coefficient of the linear term; write it down. Square it; add it to both sides of the equation: "fill in the blanks".
  - Repeat for y.
  - Find the sum of the right-hand side of the equation.

Example:

$$x^2 + y^2 - 4x - 6y + 9 = 0$$

Move the constant term to the right-hand side

Sort/organize the left-hand side, leaving blanks after each linear term

$$\underbrace{x^2 - 4x + \_\_}_{(x - \quad)^2} + \underbrace{y^2 - 6y + \_\_}_{(y - \quad)^2} = -9 + \_\_ + \_\_$$

Set up your squares

Take half the coefficient of the linear term; write it down. Square it; add it to both sides of the equation: "fill in the blanks"

$$\underbrace{x^2 - 4x + 4}_{(x-2)^2} + \underbrace{y^2 - 6y + 9}_{(y-3)^2} = -9 + 4 + 9$$

Repeat for y

Find the sum of the right hand side of the equation

$$(x - 2)^2 + (y - 3)^2 = 4$$

The center is (2, 3) and the radius is 2.

__*Examples:*__ Write the equation in standard form. Identify the coordinates of the center and the radius.

$x^2 + y^2 - 2x + 4y + 4 = 0$ Center: $(1, -2)$ Radius: 1

$(x^2 - 2x + \underline{1}) + (y^2 + 4y + \underline{4}) = -4 + 1 + 4$

$(x-1)^2 + (y+2)^2 = 1$

❖ General Form → Standard Form

- ➢ Consider the equation: $2x^2 + 2y^2 - x + 4y + 2 = 0$
  - How do the A and B values in this equation compare with those in the equation above?

    A=2 & B=2 instead of both being 1
  - How can you tell this equation represents a circle?

    A=B

- Completing the Square (A & $B > 1$)
  - Move the constant term to the right-hand side
  - Sort/organize the left-hand side
  - Factor out the coefficient of the quadratic term
  - Set up your squares
  - Take half the coefficient of the linear term; write it down. Square it.
    - What are you adding to both sides of the equation?
    - Multiply the value of c by the common factor; add this value to the right-hand side.
  - Repeat for y
  - Find the sum of the right-hand side of the equation
  - Divide all three terms by the common factor.
  - Identify the center & radius:

$$2x^2 + 2y^2 - x + 4y + 2 = 0$$

$(2x^2 - x) + (2y^2 + 4y) = -2$

$2(x^2 - 1/2x + 1/16) + 2(y^2 + 2y + 1) = -2$ $+\frac{2}{16}$ $+2$

$2(x - 1/4)^2 + 2(y+1)^2 = \frac{2}{16}$

$(x - 1/4)^2 + (y+1)^2 = \frac{1}{16}$

Center: $(1/4, -1)$ Radius: $\frac{1}{4}$

Example: Write the equation in standard form. Identify the coordinates of the center and the radius.

$$4x^2 + 4y^2 - 20x - 32y + 81 = 0$$

Center: $(5/2, 4)$ Radius: $\sqrt{2}$

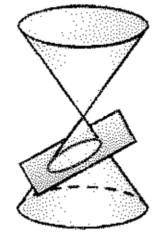
$(4x^2 - 20x) + (4y^2 - 32y) = -81$

$4(x^2 - 5x + 25/4) + 4(y^2 - 8y + 16) = -81 + 25 + 64$

$4(x - 5/2)^2 + 4(y-4)^2 = 8$

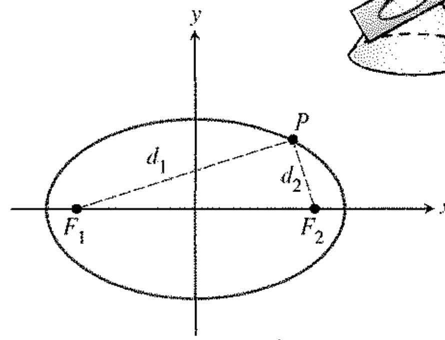
$(x - 5/2)^2 + (y-4)^2 = 2$

14.3.D1 ~ Ellipses: Graphing & Properties



OBJECTIVES:

- Identify the center, vertices, co-vertices, and foci of an ellipse and sketch its graph
- Given the equation of an ellipse in general form, complete the square to find the center, vertices, co-vertices, and foci

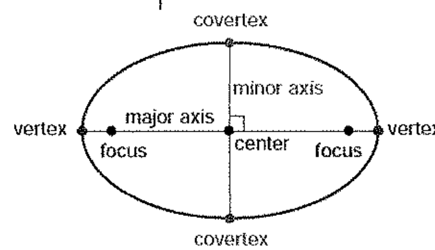


❖ Ellipse

➢ An ellipse is a set of all points P in a plane such that the sum of the distances from P to two fixed points (the foci), F_1 and F_2, is a constant k.

  ▪ $d_1 + d_2 = k$

❖ Parts of an Ellipse



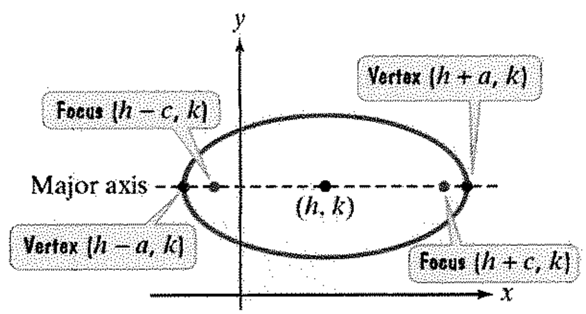
The **major axis** of an ellipse is the segment that passes through the foci, with endpoints on the ellipse; those endpoints are called **vertices**. The **minor axis** is the perpendicular bisector of the major axis; its endpoints are also on the ellipse and are called **covertices**. The intersection of the axes is the **center** of the ellipse. The center is the midpoint of both axes: Each half of the major axis is a **semimajor axis**; each half of the minor axis is a **semiminor axis**. The major axis is the longer axis.

❖ Properties of Ellipses

- Center (h, k)
- Pythagorean Relation: $c^2 = a^2 - b^2$
  - a is the distance from the center to the vertices
  - b is the distance from the center to the co-vertices
  - c is the distance from the center to the foci
- Major axis: length = 2a
- Minor axis: length = 2b

➢ Horizontal Ellipses

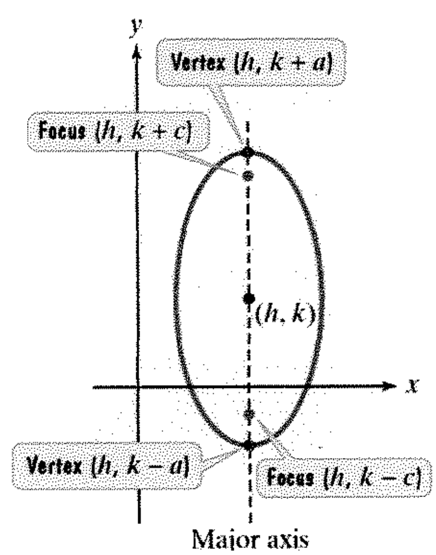
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, a > b > 0$$



- Major axis is horizontal; $y = k$
- major endpts → Vertices: $(h \pm a, k)$
- minor endpts → Co-vertices: $(h, k \pm b)$
- Foci: $(h \pm c, k)$

➢ Vertical Ellipses

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1, a > b > 0$$



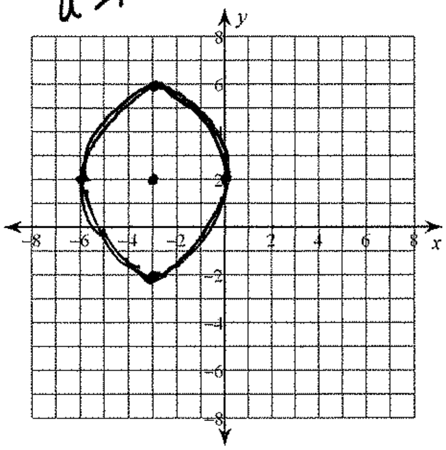
- Major axis is vertical; $x = h$
- Vertices: $(h, k \pm a)$
- Co-vertices: $(h \pm b, k)$
- Foci: $(h, k \pm c)$

Examples: Determine whether the ellipse is vertical or horizontal. Find the coordinates of the center, vertices, co-vertices, and foci of the ellipse with the given equation. Then sketch its graph.

1. $\frac{(x+3)^2}{9} + \frac{(y-2)^2}{16} = 1$

$b^2 \nearrow 9$ $a^2 \nearrow 16$

$a=4$
$b=3$
$c^2=16-9$
$c^2=7$
$c=\sqrt{7}$



VERTICAL OR HORIZONTAL?

CENTER: $(-3,2)$

VERTICES: $(h, k \pm a) = (-3, 2 \pm 4)$
(Need a)
$(-3,6)$ & $(-3,-2)$

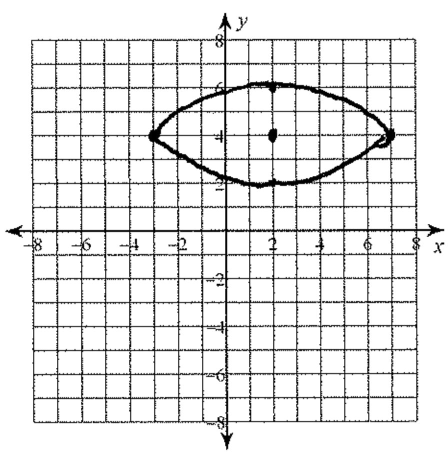
CO-VERTICES: $(h \pm b, k) = (-3 \pm 3, 2)$
(Need b)
$(0,2)$ & $(-6,2)$

FOCI: $(h, k \pm c) = (-3, 2 \pm \sqrt{7})$
(Need c)

2. $\frac{(x-2)^2}{25} + \frac{(y-4)^2}{4} = 1$

$a^2 \nearrow 25$

$a=5$
$b=2$
$c^2=25-4$
$c^2=21$
$c=\sqrt{21}$



VERTICAL OR HORIZONTAL?

CENTER: $(2,4)$

VERTICES: $(h \pm a, k) = (2 \pm 5, 4)$
(Need a)
$(7,4)$ & $(-3,4)$

CO-VERTICES: $(h, k \pm b) = (2, 4 \pm 2)$
(Need b)
$(2,6)$ & $(2,2)$

FOCI: $(h \pm c, k)$
(Need c)
$(2 \pm \sqrt{21}, 4)$

- General Form Equation of an Ellipse
  - $Ax^2 + By^2 + Cx + Dy + E = 0$, where A, B, C, and D are constants, $A \neq B$, and $x \neq y$.
    - In order to identify the center, vertices, co-vertices and foci of an ellipse written in general form, it is necessary to rewrite the equation in standard form.

- General Form → Standard Form
  - Completing the Square
    - Follow the completing the square process for a circle when A & $B > 1$ (see page 3)
    - Divide both sides of the equation by the constant term on the right-hand side.

Examples: Write the equation in standard form. Determine whether the ellipse is vertical or horizontal. Find the coordinates of the center, vertices, co-vertices, and foci of the ellipse with the given equation.

3. $x^2 + 4y^2 + 14x - 32y + 49 = 0$

$(x^2+14x)+(4y^2-32y)=-49$

$(x^2+14x+\underline{49})+4(y^2-8y+\underline{16})=-49+49+64$

$\frac{(x+7)^2}{64}+\frac{\cancel{4}(y-4)^2}{\cancel{16}\,64}=\frac{\cancel{64}}{\cancel{64}}\ 1$

$a^2=64 \qquad b^2=16$

$a=8 \qquad b=4$

$c^2=64-16=48$

$c=\sqrt{48}=4\sqrt{3}$

STANDARD FORM: $\frac{(x+7)^2}{64}+\frac{(y-4)^2}{16}=1$

VERTICAL OR HORIZONTAL? HORIZONTAL

CENTER: $(-7,4)$

VERTICES: $(h\pm a,k)=(-7\pm 8,4)$
(Need a)
$(1,4)$ & $(-15,4)$

CO-VERTICES: $(h,k\pm b)=(-7,4\pm 4)$
(Need b)
$(-7,8)$ & $(-7,0)$

FOCI: $(h\pm c,k)$
(Need c)
$(-7\pm 4\sqrt{3},4)$

4. $16x^2 + 9y^2 - 160x + 18y - 167 = 0$

$(16x^2-160x)+(9y^2+18y)=167$

$16(x^2-10x+\underline{25})+9(y^2+2y+\underline{1})=167+400+9$

$\frac{\cancel{16}(x-5)^2}{\cancel{576}\,36}+\frac{\cancel{9}(y+1)^2}{\cancel{576}\,64}=\frac{\cancel{576}}{\cancel{576}}\ 1$

$a^2=64 \qquad b^2=36$

$a=8 \qquad b=6$

$c^2=64-36=28$

$c=\sqrt{28}=2\sqrt{7}$

STANDARD FORM: $\frac{(x-5)^2}{36}+\frac{(y+1)^2}{64}=1$

VERTICAL OR HORIZONTAL? VERTICAL

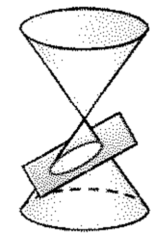
CENTER: $(5,-1)$

VERTICES: $(h,k\pm a)=(5,-1\pm 8)$
(Need a)
$(5,7)$ & $(5,-9)$

CO-VERTICES: $(h\pm b,k)=(5\pm 6,-1)$
(Need b)
$(11,-1)$ & $(-1,-1)$

FOCI: $(h,k\pm c)$
(Need c)
$(5,-1\pm 2\sqrt{7})$

14.3.D2 ~ Ellipses: Writing Equations

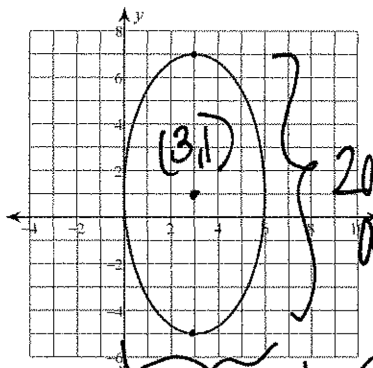


OBJECTIVE:

- Write the standard form equation of an ellipse given its center, the vertices, co-vertices, the foci or its graph

Write the standard form equation of the ellipse whose graph is shown.

1.



(3,1)

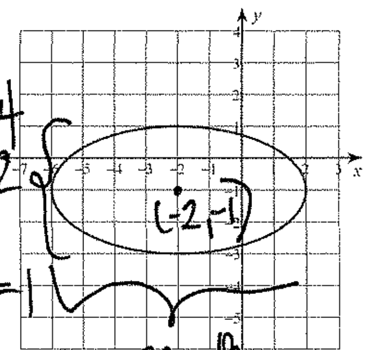
$2a=12$

$a=6$

$2b=6 \quad b=3$

$$\frac{(x-3)^2}{9}+\frac{(y-1)^2}{36}=1$$

2.



$2b=4$

$b=2$

(-2,-1)

$2a=8$

$a=4$

$$\frac{(x+2)^2}{16}+\frac{(y+1)^2}{4}=1$$

Write the equation of the ellipse with the following characteristics.

3. Center: $(-5, 9)$
   Vertex: $(9, 9)$
   Focus: $(-5-4\sqrt{6}, 9)$

Vertical or horizontal?
Need to know: the center and the values of a & b.

$h+a=9$

$-5+a=9$

$a=14$

$a^2=196$

$h-c=-5-4\sqrt{6}$

$c=4\sqrt{6}$

$c^2=96$

$c^2=a^2-b^2$

$96=196-b^2 \rightarrow +100=b^2$

$$\frac{(x+5)^2}{196}+\frac{(y-9)^2}{100}=1$$

4. Foci: $(12,-10)$, $(6,-10)$
   Co-vertices: $(9,-6)$, $(9,-14)$

Vertical or horizontal?
Need to know: the center and the values of a & b.

CTR $(9,-10)$

$h+c=12$

$9+c=12$

$c=3$

$c^2=9$

$k+b=-6$

$-10+b=-6$

$b=4$

$b^2=16$

$$\frac{(x-9)^2}{25}+\frac{(y+10)^2}{16}=1$$

$c^2=a^2-b^2$

$9=a^2-16$

$25=a^2$

5. Foci: (−4, 15), (−4, 3)
Endpoints of major axis: (−4, 19), (−4, −1)

aka vertices

major axis = 20 = 2a
10 = a
100 = a^2

19 = k + a
19 = k + 10
9 = k

15 = k + c
15 = 9 + c
6 = c
36 = c^2

$c^2 = a^2 - b^2$
$36 = 100 - b^2$
$+64 = +b^2$

Vertical or horizontal?
Need to know: the center and the values of a & b.

ctr (−4, 9)

$$\frac{(x+4)^2}{64} + \frac{(y-9)^2}{100} = 1$$

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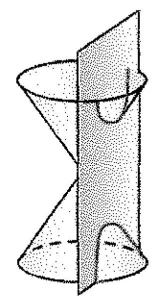
14.4.D1 ~ Hyperbolas: Graphing & Properties



OBJECTIVES:

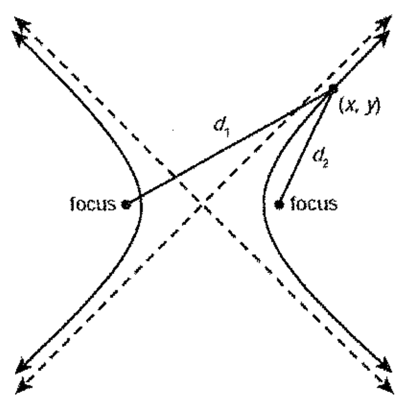
- Identify the center, vertices, and foci of a hyperbola & sketch its graph
- Given the equation of a hyperbola in general form, complete the square to find the center, vertices, foci, and asymptotes

❖ Hyperbola

➢ A hyperbola is the set of all points P in a plane such that the absolute value of the difference between the distances from P to two fixed points (the foci), F_1 and F_2, is a constant k.

❖ Parts of a Hyperbola

Every hyperbola has two separate **branches**. The line through the foci intersects the hyperbola at two **vertices**. The **transverse axis** is the line segment joining the vertices. The midpoint of the transverse axis is the **center** of the hyperbola. Every hyperbola has an **associated rectangle**. The diagonals of the associated rectangle pass through the center. The lines containing these diagonals are **asymptotes** of the hyperbola. As a point in the hyperbola gets farther from either vertex, it approaches—but never touches—one of the asymptotes. The **conjugate axis** is the line segment through the center, perpendicular to the transverse axis, with endpoints on the associated rectangle. A hyperbola is symmetric with respect to both its transverse axis and its conjugate axis.



$|d_2 - d_1|$ is constant, no matter where (x, y) is on the hyperbola.
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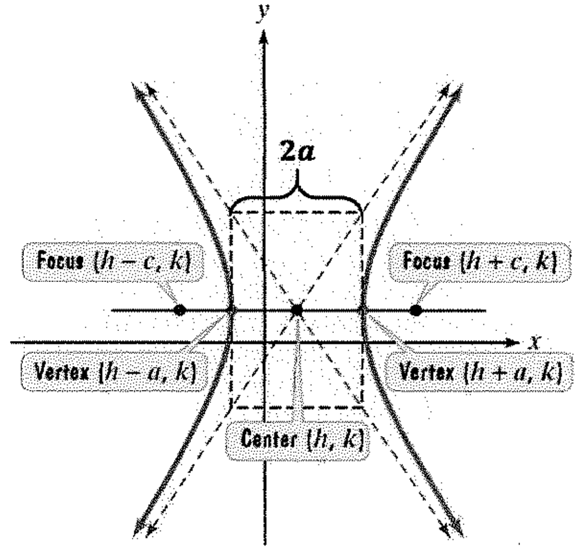
- Properties of Hyperbolas
  - Center (h, k)
  - Pythagorean Relation: $c^2 = a^2 + b^2$
    - a is the distance from the center to the vertices
    - b is the distance from the vertices to the corners of the associated rectangle
    - c is the distance from the center to the foci
  - The asymptotes pass through the corners of the associated rectangle with dimensions $2a \times 2b$
    - The line segment of length $2b$ is the conjugate axis and is perpendicular to the transverse axis through the center of the hyperbola.

- Horizontal Hyperbolas

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

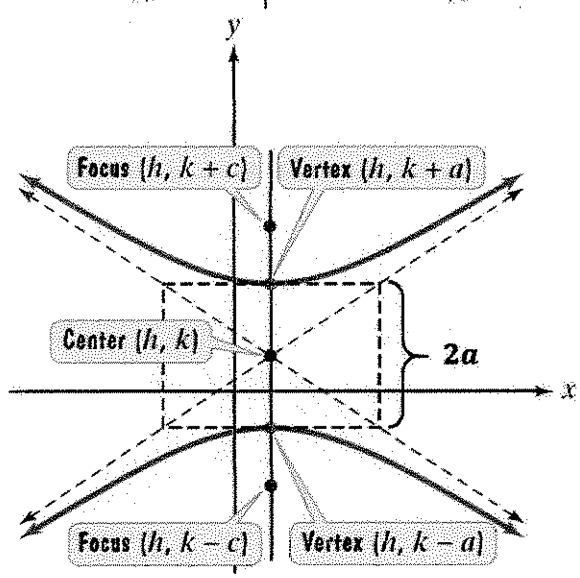
  - Transverse axis: horizontal; length = $2a$
  - Vertices: $(h \pm a, k)$ transverse axis endpts
  - Foci: $(h \pm c, k)$
  - Asymptotes: $y - k = \pm \frac{b}{a}(x - h)$



- Vertical Hyperbolas

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

  - Transverse axis: vertical; length = $2a$
  - Vertices: $(h, k \pm a)$
  - Foci: $(h, k \pm c)$
  - Asymptotes: $y - k = \pm \frac{a}{b}(x - h)$



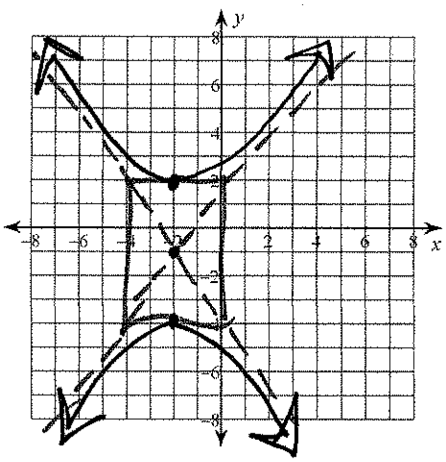
- Graphing a Hyperbola
  - Locate and plot the vertices.
  - Draw the associated rectangle with dimensions $2a \times 2b$
  - Draw extended diagonals of the rectangle to obtain the asymptotes.
  - Draw the two branches of the hyperbola by starting at each vertex and approaching the asymptotes.

Example: Determine whether the hyperbola is vertical or horizontal. Find the coordinates of the center, vertices, and foci of the hyperbola with the given equation. Then sketch its graph.

1. $\frac{(y+1)^2}{9} - \frac{(x+2)^2}{4} = 1$

$a^2 \nearrow$

$a = 3$
$b = 4$
$c^2 = 9 + 4$
$c^2 = 13$
$c = \sqrt{13}$



VERTICAL OR HORIZONTAL?

CENTER: $(-2, -1)$

VERTICES: (Need a) $(h, k \pm a) = (-2, -1 \pm 3)$
$(-2, 2)$ & $(-2, -4)$

FOCI: (Need c) $(h, k \pm c) = (-2, -1 \pm \sqrt{13})$

- General Form Equation of a Hyperbola
  - $Ax^2 + By^2 + Cx + Dy + E = 0$, where $A, B, C,$ and D are constants, A & B have opposite signs, and $x \neq y$.
    - In order to identify the center, vertices, foci, and asymptotes of a hyperbola written in general form, it is necessary to rewrite the equation in standard form.

- General Form → Standard Form
  - Completing the Square
    - Follow the completing the square process for a circle when A & $B > 1$ (see page 3)
    - Divide both sides of the equation by the constant term on the right-hand side.
    - Pay attention to the order because subtraction is involved; positive quantity comes first.

Example: Write the equation in standard form. Find the coordinates of the center, vertices, & foci of the hyperbola with the given equation.

2. $-4x^2 + y^2 - 48x - 12y - 252 = 0$

$(-4x^2 - 48x) + (y^2 - 12y) = 252$
$-4(x^2 + 12x + 36) + (y^2 - 12y + 36) = 252$
-144
$+36$
$\frac{-\cancel{4}(x+6)^2}{\cancel{144}\ 36} + \frac{(y-6)^2}{144} = \frac{\cancel{144}}{\cancel{144}}$

$a^2 = 144$
$a = 12$

$b^2 = 36$
$c^2 = a^2 + b^2$
$c^2 = 144 + 36$
$c^2 = 180$ $c = \sqrt{180} = 6\sqrt{5}$

STANDARD FORM: $\frac{(y-6)^2}{144} - \frac{(x+6)^2}{36} = 1$

VERTICAL OR HORIZONTAL?

CENTER: $(-6, 6)$

VERTICES: (Need a) $(h, k \pm a) = (-6, 6 \pm 12)$
$(-6, 18)$ & $(-6, -6)$

FOCI: (Need c) $(h, k \pm c) = (-6, 6 \pm 6\sqrt{5})$

3. $9x^2 - 16y^2 - 36x - 64y + 116 = 0$

$(9x^2 - 36x) + (-16y^2 - 64y) = -116$

$9(x^2 - 4x + 4) - 16(y^2 + 4y + 4) = -116 + 36 - 64$

$\dfrac{-9(x-2)^2}{-144} + \dfrac{16(y+2)^2}{-144} = \dfrac{-144}{-144}$

$\dfrac{(x-2)^2}{16} \quad \dfrac{(y+2)^2}{9} \quad 1$

$a^2 = 9$
$a = 3$

$c^2 = a^2 + b^2$
$c^2 = 9 + 16 = 25$
$c = 5$

STANDARD FORM: $\dfrac{(y+2)^2}{9} - \dfrac{(x-2)^2}{16} = 1$

VERTICAL OR HORIZONTAL?

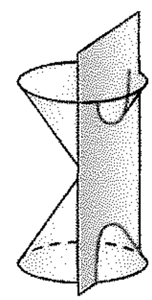
CENTER: $(2,-2)$

VERTICES: $(h, k \pm a) = (2, -2 \pm 3)$
(Need a) $(2,1)$ & $(2,-5)$

FOCI: $(h, k \pm c) = (2, -2 \pm 5)$
(Need c) $(2,3)$ & $(2,-7)$

~~~~~~

14.4.D2 ~ Hyperbolas: Writing Equations



OBJECTIVE:

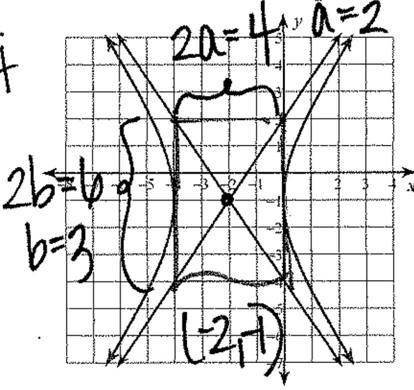
- Write the standard form equation of a hyperbola given: its center, the foci, vertices or its graph

Write the standard form equation of the hyperbola whose graph is shown.

1. H

$2a = 4$, $a = 2$

$2b = 6$, $b = 3$



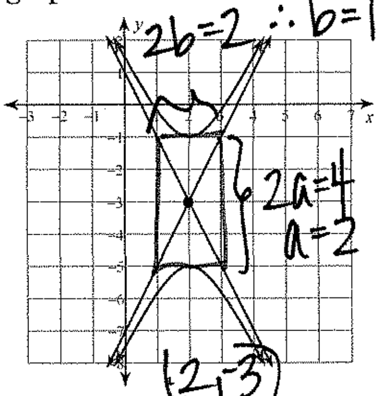
$(-2,-1)$

$\dfrac{(x+2)^2}{4} - \dfrac{(y+1)^2}{9} = 1$

2. V

$2b = 2 \therefore b = 1$

$2a = 4$, $a = 2$



$(2,-3)$

$\dfrac{(y+3)^2}{4} - \dfrac{(x-2)^2}{1} = 1$

Write the equation of the hyperbola with the following characteristics.

3. Center: $(4, 9)$
Transverse axis is vertical and 18 units long
Conjugate axis is 20 units long

$2a = 18$
$a = 9$

$2b = 20$
$b = 10$

Vertical or horizontal?
Need to know: the center and the values of a & b.

$\dfrac{(y-9)^2}{81} - \dfrac{(x+4)^2}{100} = 1$
~~~~~~

4. Vertices: $(9, -2), (-3, -2)$
Distance from center to focus = $\sqrt{205} = c$

Vertical or horizontal?
Need to know: the center and the values of a & b.

$9 = h + a \rightarrow 9 = 3 + a$
$6 = a$

$+ \quad -3 = h - a$

$6 = 2h$
$3 = h$

$c^2 = a^2 + b^2$
$205 = 36 + b^2$
$169 = b^2$

CTR $(3, -2)$

$$\frac{(x-3)^2}{36} - \frac{(y+2)^2}{169} = 1$$

5. Vertices: $(4, 10), (4, -4)$
Endpoints of conjugate axis: $(10, 3), (-2, 3)$

Vertical or horizontal?
Need to know: the center and the values of a & b.

length $= 2b = 12$
$b = 6$

$k + a = 10$
$3 + a = 10$
$a = 7$

CTR $= (4, 3)$

$$\frac{(y-3)^2}{49} - \frac{(x-4)^2}{36} = 1$$

6. Foci: $(3, 11), (3, -19)$
Vertices: $(3, 8), (3, -16)$

Vertical or horizontal?
Need to know: the center and the values of a & b.

$8 = k + a \rightarrow 8 = -4 + a$
$12 = a$

$+ \quad -16 = k - a$

$-8 = 2k$
$-4 = k$

$11 = k + c$
$11 = -4 + c$
$15 = c$

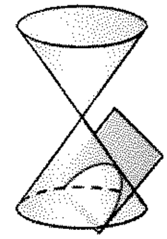
$c^2 = a^2 + b^2$
$225 = 144 + b^2$
$81 = b^2$

CTR: $(3, -4)$

$$\frac{(y+4)^2}{144} + \frac{(x-3)^2}{81} = 1$$

$225 = c^2$

14.5D1 ~ Parabolas: Graphing & Properties



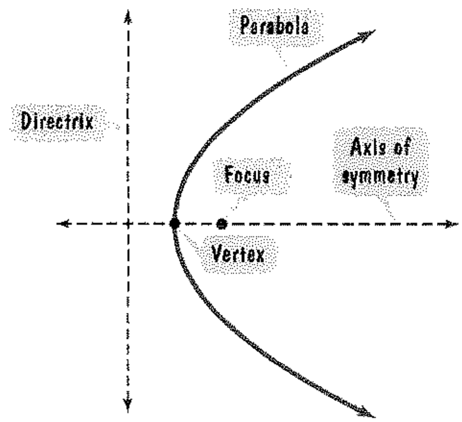
OBJECTIVES:

- Identify the vertex, the focus, and the directrix of a parabola & sketch its graph
- Given the equation of a parabola in general form, complete the square to find the vertex, focus, and the directrix

❖ Parabola & its Parts

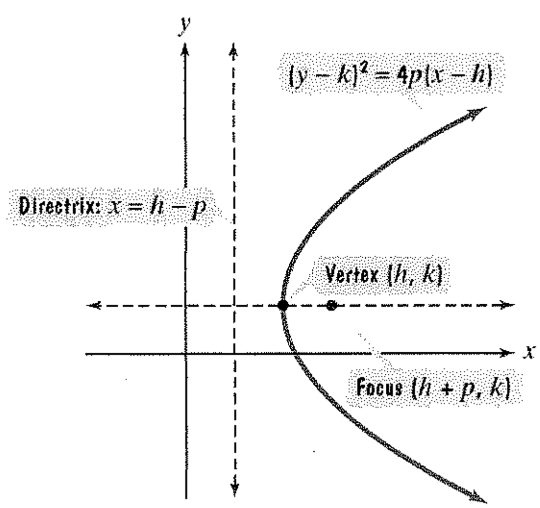
➢ A **parabola** is the set of all points in a plane that are equidistant from a fixed point (the *focus*) and a fixed line (the *directrix*).

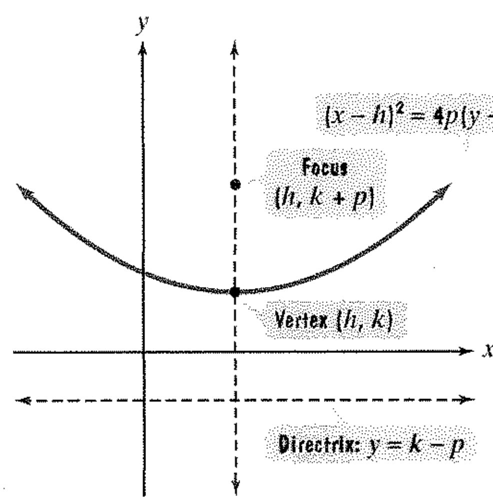
- The *axis of symmetry* passes through the focus & is perpendicular to the directrix.
- The *vertex* is the point of intersection of the parabola with its axis of symmetry and is halfway between the focus and the directrix.
- The *latus rectum* of a parabola is a line segment that passes through its focus, is parallel to its directrix, and has its endpoints on the parabola.
  - The length of the latus rectum is $|4p|$.
  - Endpoints of the latus rectum are helpful in determining a parabola's "width" or how it opens.



❖ Properties of Parabolas

Equation	Vertex	Focus	Directrix	Opens
$(x-h)^2 = 4p(y-k)$	(h,k)	$(h,k+p)$	$y=k-p$	UP
$(x-h)^2 = -4p(y-k)$	(h,k)	$(h,k-p)$	$y=k+p$	DOWN
$(y-k)^2 = 4p(x-h)$	(h,k)	$(h+p,k)$	$x=h-p$	RIGHT
$(y-k)^2 = -4p(x-h)$	(h,k)	$(h-p,k)$	$x=h+p$	LEFT

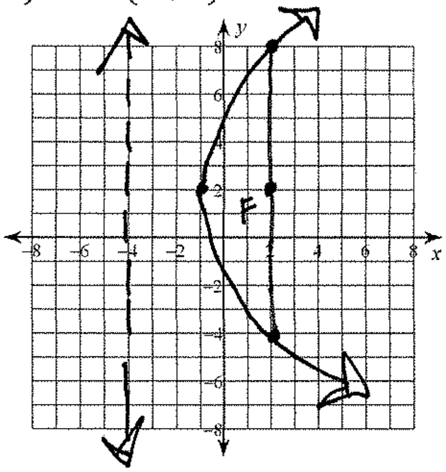




Example: Determine the direction of opening, the coordinates of the vertex, the focus, the equation of the directrix, and the coordinates of the endpoints of the latus rectum of the parabola. Then sketch its graph.

1. $(y-2)^2 = 12(x+1)$

$12 = 4p$
$3 = p$



DIRECTION OF OPENING: Right

VERTEX $(-1, 2)$

FOCUS: $(h+p, k) = (-1+3, 2) = (2, 2)$

DIRECTRIX: $x = h - p \rightarrow x = -1-3 \rightarrow x = -4$

ENDPOINTS OF THE LATUS RECTUM:

length = 12 $\quad (2, 8)$ & $(2, -4)$

- General Form → Standard Form
  - Completing the Square ($a \neq 1$)

Example:

$$x = -3y^2 + 12y - 13$$

Sort/organize the right-hand side, leaving a positive blank after the linear term and a negative blank after the constant term.

$$x = -3y^2 + 12y + \_\_\_ - 13 - \_\_\_$$

Factor out the coefficient (of the quadratic term) from the first two terms.

$$x = \underbrace{-3(y^2 - 4y + \_\_\_)}_{} - 13 - \_\_\_$$

Set up your square.

$$x = -3(y - \quad)^2$$

Take half the coefficient of the linear term; write it down. Square it; add it after the linear term.

Multiply it by the common factor and subtract the product from the constant term.

$$x = \underbrace{-3(y^2 - 4y + 4)}_{} - 13 - (-12)$$

Simplify.

$$x = -3(y-2)^2 - 1$$

Write in standard form.

$$(y-2)^2 = -\frac{1}{3}(x+1)$$

Examples: Write the equation in standard form. Determine whether the parabola opens up, down, right, and left. Identify the coordinates of the vertex, the focus, and the equation of the directrix of the parabola with the given equation.

2. $y = (7x^2 - 28x) + 32$

$y = 7(x^2 - 4x + 4) + 32 - 28$

$y = 7(x-2)^2 + 4$

$\frac{1}{7}(y-4) = \frac{1}{7} \cdot 7(x-2)^2$

$\frac{1}{7} = 4p$

$1 = 28p \qquad p = 1/28$

STANDARD FORM: $(x-2)^2 = \frac{1}{7}(y-4)$

DIRECTION OF OPENING: up

VERTEX: $(2, 4)$

FOCUS: $(h, k+p) = (2, 4 + 1/28) = (2, 4\frac{1}{28})$

DIRECTRIX: $y = k - p$

$y = 4 - 1/28 \qquad y = 3\,27/28$

- General Conic Form of a Parabola
  - $Ax^2 + By^2 + Cx + Dy + E = 0$, where A, B, C, and D are constants, $A = 0$ or $B = 0$, but not both, and $x \neq y$.

Example: Write the equation in standard form. Determine the direction of opening, the coordinates of the vertex, the focus, the equation of the directrix, and the coordinates of the endpoints of the latus rectum of the parabola. Then sketch its graph.

3. $-y^2 + 4x + 2y + 23 = 0$

$4x = (y^2 - 2y + 1) - 23 - 1$

$4x = (y-1)^2 - 24$

$4x + 24 = (y-1)^2$

$4(x+6) = (y-1)^2$

$4p = 4$

$p = 1$

STANDARD FORM: $(y-1)^2 = 4(x+6)$

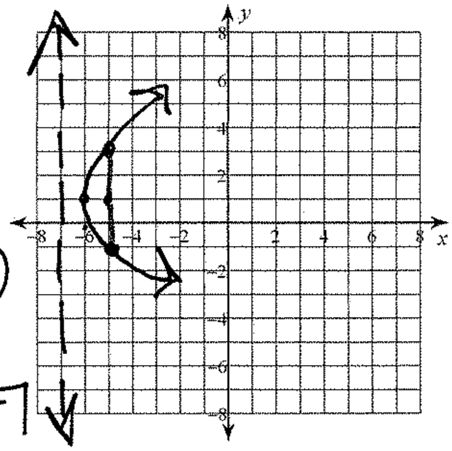
DIRECTION OF OPENING: Right

VERTEX: $(-6, 1)$

FOCUS: $(h+p, k) = (-6+1, 1)$ $(-5, 1)$

DIRECTRIX: $x = -6 - 1 \rightarrow x = -7$

ENDPOINTS OF THE LATUS RECTUM: length $= 4$ $(-5, 3)$ & $(-5, -1)$



4. $-2x^2 + 20x + y - 48 = 0$

$-2x^2 + 20x - 48 = -y$

$-2(x^2 - 10x + 25) - 48 - 50 = -y$

$-2(x-5)^2 + 2 = -y$

$+2(x-5)^2 = +y + 2$

$(x-5)^2 = \frac{1}{2}(y+2)$

$\frac{1}{2} = 4p$

$1 = 8p$

$\frac{1}{8} = p$

STANDARD FORM: $(x-5)^2 = \frac{1}{2}(y+2)$

DIRECTION OF OPENING: up

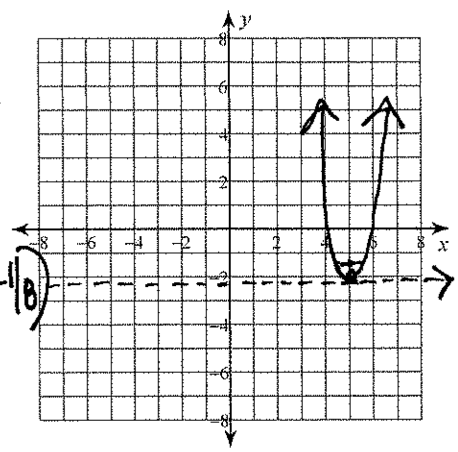
VERTEX: $(5, -2)$

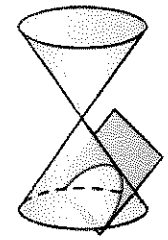
FOCUS: $(h, k+p) = (5, -2 + 1/8)$ $(5, -1\,7/8)$

DIRECTRIX: $y = -2\,1/8$

ENDPOINTS OF THE LATUS RECTUM:

length $= 1/2$





14.5D2 ~ Parabolas: Writing Equations

OBJECTIVE:

- Write the standard form equation of a parabola given: the vertex, focus, or directrix

Write the equation of the parabola with the following characteristics.

1. Vertex: $(-3, -3)$
   Focus: $\left(-\frac{25}{8}, -3\right)$

Rt or left

$h+p = \frac{-25}{8}$

$-3+p = \frac{-25}{8}$

$p = \frac{-1}{8} \times 4 = \frac{-1}{2}$

$(y+3)^2 = \frac{-1}{2}(x+3)$

F

2. Vertex: $(1, -3)$
   Directrix: $x = 4$ → $4 = h+p$

Rt or left

4

$4 = 1+p$

$3 = p$

$\times 4$

12

$(y+3)^2 = 12(x-1)$

3. Focus: $(-10, -9)$
   Directrix: $y = -11$

up or down

$h = -10$

-9

F

-11

$k+p = -9$ → $-10+p = -9$

$+\ k-p = -11$

$2k = -20$

$k = -10$

$p = 1$

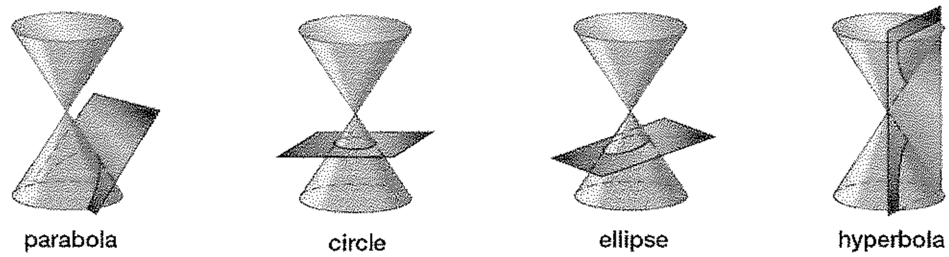
$(x+10)^2 = 4(y+10)^2$

14.6 ~ Classifying Conics

OBJECTIVES:

- Classify/identify a conic section from its equation in general conic form
- Write the standard form equation of a circle, parabola, ellipse, or hyperbola given its equation in general form and identify the center, radius (for a circle), vertices, and foci

Recall that parabolas, circles, ellipses, and hyperbolas are called conic sections because they are the cross sections formed when a double cone is sliced by a plane. You can use a flashlight and a flat surface to make patterns in the shapes of conic sections.



❖ General Conic Form

- $Ax^2 + By^2 + Cx + Dy + E = 0$, where A, B, C, and D are constants, and $x \neq y$
- The relationship between A and B is enough to determine whether the conic section is a circle, parabola, ellipse, or hyperbola.
  - What is the relationship between A and B in a...
    - Circle: $A = B$
    - Parabola: $A = 0$ or $B = 0$
    - Ellipse: $A \neq B$
    - Hyperbola: A ÷ B have opposite signs

Examples: Identify the conic section as a circle, parabola, ellipse, or hyperbola and explain your reasoning.

1. $4x^2 + y^2 - 16x - 6y + 9 = 0$

   $A \neq B$ ellipse

2. $3y^2 - x - 6y + 5 = 0$

   parabola
   $A = 0$

3. $2y^2 - 3x^2 + 4x + 12y - 22 = 0$

   hyperbola opp. signs

4. $3y^2 + 3x^2 + x - 4y - 9 = 0$

   circle $A = B$

5. $9x^2 + 3x + 10 = 16y^2 + 154 + 3x$

   $-16y^2$
   hyperbola
   opp. signs

6. $-x^2 - 8x - y + 19 = 0$

   parabola

Classify each conic section and complete the square to write its equation in standard form. For circles, identify the center and radius. For parabolas, identify the vertex and focus. For ellipses and hyperbolas identify the center, vertices, and foci.

7. $4x^2 - y^2 - 16x - 4 = 0$

$(4x^2 - 16x) - y^2 = 4$

$4(x^2 - 4x + 4) - y^2 = 4 + 16$

$\frac{4(x-2)^2}{20} - \frac{y^2}{20} = \frac{20}{20}$

$\frac{(x-2)^2}{5} - \frac{y^2}{20} = 1$

$c^2 = 5 + 20 = 25$
$c = 5$

CONIC SECTION: hyperbola

CENTER: $(2,0)$

RADIUS: N/A

VERTEX/VERTICES: $(h \pm a, k) = (2 \pm \sqrt{5}, 0)$

FOCUS/FOCI: $(h \pm c, k) = (2 \pm 5, 0)$
$(7,0)$ & $(-3,0)$

8. $2x^2 + 16x + 3y + 38 = 0$

$3y = -2x^2 - 16x - 38$

$3y = -2(x^2 + 8x + 16) - 38 + \underline{32}$

$3y = -2(x+4)^2 - 6$

$3y + 6 = -2(x+4)^2$

$3(y+2) = -2(x+4)^2$

$(x+4)^2 = -3/2(y+2)$

$\frac{-3}{2} = -4p$
$-3 = 8p$
$p = +\frac{3}{8}$

CONIC SECTION: parabola

~~CENTER:~~

~~RADIUS:~~

VERTEX/VERTICES: $(-4,-2)$

FOCUS/FOCI: $(h, k-p) = (-4, -2 - 3/8)$
$(-4, -2\,3/8)$

9. $16x^2 + 9y^2 + 128x - 54y + 193 = 0$

$(16x^2 + 128x) + (9y^2 - 54y) = -193$

$16(x^2 + 8x + 16) + 9(y^2 - 6y + 9) = -193 + 256 + 81$

$\frac{16(x+4)^2}{144} + \frac{9(y-3)^2}{144} = \frac{144}{144}$

$\frac{(x+4)^2}{9} + \frac{(y-3)^2}{16} = 1$
$\quad b^2 \qquad a^2$

$c^2 = 16 - 9 = 7$
$c = \sqrt{7}$

CONIC SECTION: ellipse

CENTER: $(-4,3)$

~~RADIUS:~~

VERTEX/VERTICES: $(h, k \pm a) = (-4, 3 \pm 4)$
$(-4,7)$ & $(-4,-1)$

FOCUS/FOCI: $(h, k \pm c) = (-4, 3 \pm \sqrt{7})$

10. $4x^2 + 4y^2 + 20x + 8y - 7 = 0$

$(4x^2+20x)+(4y^2+8y)=7$

$4(x^2+5x+\frac{25}{4})+4(y^2+2y+1)=7+25+4$

$\frac{4(x+5/2)^2}{4}+\frac{4(y+1)^2}{4}=\frac{36}{4}$

$(x+5/2)^2+(y+1)^2=9$

CONIC SECTION: Circle

CENTER: $(-5/2, -1)$

RADIUS: 3

~~VERTEX/VERTICES:~~

~~FOCUS/FOCI:~~