CHAPTER 1: MATHEMATICAL MODELING, FUNCTIONS, & CHANGE

1.2.D1 ~ Functions & Function Notation

Objectives:

- Identify a functional relationship between two variables
- Identify input/independent & output/dependent variables in situations involving two variable quantities
- Evaluate & interpret functions using function notation
- Solve function equations for a given variable

Algebra is the branch of mathematics that deals with variables and the relationships between and among variables. A variable is a quantity that may change in value from one particular instance to another.

❖ Two Variable Quantities

- ➢ The input is the value that is given first in an input/output relationship.
- ➢ The output is the second number in an input/output situation. It is the number that is associated with or results from the input value.

❖ What is a Function?

- ➢ A function is a correspondence between an input variable and an output variable that assigns a single output value to each input value.
  - ▪ Any given input value has exactly one corresponding output value.
- ➢ A functional relationship is stated as follows: "The output variable is a function of the input variable."
- ➢ If the relationship between two variables is a function, the input variable is called the independent variable, and the output variable is called the dependent variable.

EXAMPLE:

1. Which of the following relationships represents a function? Explain your reasoning.

a.

Input, x	0	1	2	3	4
Output, y	8	8	8	8	8

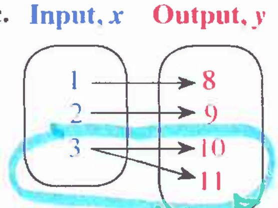
Function!

b.

Input, x	8	8	8	8	8
Output, y	0	1	2	3	4

Not the input of 8 has 5 outputs

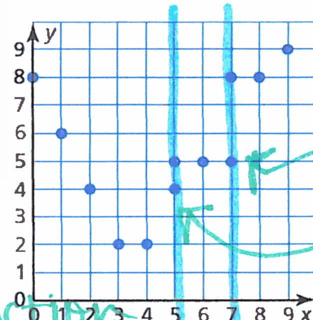
c. Input, x Output, y



NOT

two outputs

d.



NOT

input of 5 & 7 have 2 outputs!

e. $(-2, 5), (-1, 8), (0, 6), (1, 6), (2, 7)$ Function

f. $(-2, 0), (-1, 0), (-1, 1), (0, 1), (1, 2), (2, 2)$ NOT!

g. Each radio frequency x in a listening area has exactly one radio station y. Function!

h. The same television station x can be found on more than one channel y.

NOT!

❖ Function Notation

➢ $f(x)$ represents the output value corresponding to a given x; x is the input variable

- If y represents the output variable, f is the name of the function, and x represents the input variable, then $y = f(x)$ is read "y equals f of x"

EXAMPLE:

2. If you work for an hourly wage, your gross pay is a function of the number of hours you work.

 a. Identify the independent variable and the dependent variable.

 hours worked gross pay

 b. If you earn $9.50 per hour, complete the following table:

Number of hours	0	3	5	7	10	12
Gross pay	0	28.50	47.50	66.50	95	114

 c. Write a sentence explaining the meaning of $f(10) = 95$.

 For 10 hours worked, the gross pay is \$95.

❖ Evaluating Functions

➢ The process of finding the *output* of a function that corresponds with a given *input* is called evaluating a function.

- Evaluate $f(642)$ means "find the output value that corresponds with an input value of 642."

❖ Solving Function Equations

➢ The process of finding the *input* of a function that corresponds with a given *output* is called solving a function equation.

EXAMPLES:

Evaluate the function for the specified value. Note that special symbols do not have any special mathematical meaning. They are just symbols.

3. $t(v) = -v^2 + 3v - \frac{4}{v};\ t(-4)$

$= -(-4)^2 + 3(-4) - \frac{4}{-4}$

$= -16 + (-12) - (-1) = -27$

4. $m(x) = \sqrt{x^2 - 4x};\ m(-3\star)$

$= \sqrt{(-3\star)^2 - 4(-3\star)}$

$= \sqrt{9\star^2 + 12\star}$

5. The median sales price of new homes since 1980 can be modeled by the formula

$$m(t) = 5845t + 56{,}590 \text{ dollars}$$

Where m is the median sales price and t is the year since 1980. (Source: www.census.gov)

Solve $m(t) = 302{,}080$ for t and explain what the numerical answer represents in its real-world context.

$5845t + 56590 = 302080$

$5845t = 245490$

$t = 42$

$m(42) = 302080$

In 2022 the median sales price of a new home is \$302080.

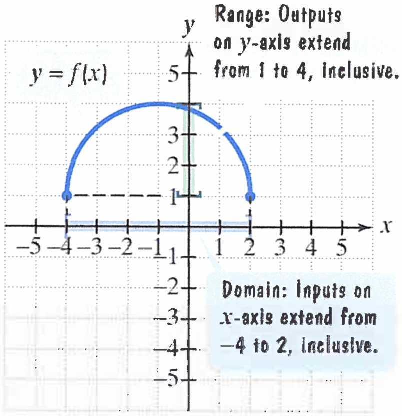
1.2.D2 ~ Analyzing Functions

Objectives:
- Determine a function's domain and range
- Calculate & interpret the meaning of an average rate of change
- Determine if a data table represents a function

❖ Domain & Range

- Domain – the collection of all possible replacement values of the input variable
- Range – the collection of all possible replacement values of the output variable

- Using a Function's Graph



x-values
DOMAIN:

$[-4, 2]$.
Left Right

The square brackets indicate -4 and 2 are included. Note the square brackets on the x-axis in **Figure 1.25**.

y-values
RANGE:

$[1, 4]$.
Low High

The square brackets indicate 1 and 4 are included. Note the square brackets on the y-axis in **Figure 1.25**.

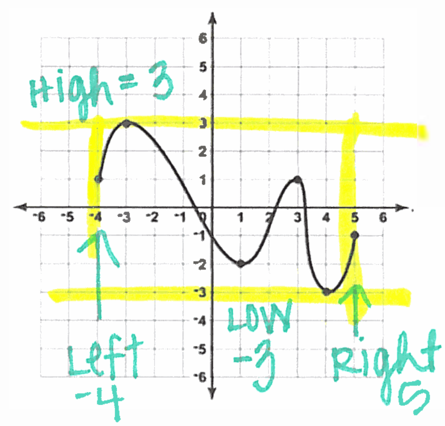
Left/Down Forever: $(-\infty, \_\_$

Right/Up Forever: $\_\_, \infty)$

Parentheses: (not solid)

Brackets: [solid]

EXAMPLES:



1. Describe the domain and range of the function for the graph shown.

   $D = [-4, 5]$ $R = [-3, 3]$

2. Which relation (below) is a function? Identify its domain and range.

   a. $\{(-1, 3), (\cancel{-2}, 6), (0, 0), (\cancel{-2}, -2)\}$
   b. $\{(-2, -2), (0, 0), (1, 1), (2, 2)\}$
   c. $\{(4, 0), (4, 1), (4, 2), (4, 3)\}$
   d. $\{(-1, 8), (-2, 8), (0, 8), (-2, -2)\}$

   $D = \{-2, 0, 1, 2\}$
   $R = \{-2, 0, 1, 2\}$

3. Describe the range of the function $f(x) = x^2 + 2x$, if its domain is $\{0, 2, 4\}$.

   $f(0) = 0^2 + 2(0) = 0$ $f(2) = 2^2 + 2(2) = 8$ $f(4) = 4^2 + 2(4) = 24$
   $R = \{0, 8, 24\}$

4. Given the function $g(x) = 2x - 1$ and its range described by the set $\{-3, 1, 5\}$, what is the domain?

   $2x - 1 = -3$ $2x - 1 = 1$ $2x - 1 = 5$
   $2x = -2$ $2x = 2$ $2x = 6$
   $x = -1$ $x = 1$ $x = 3$

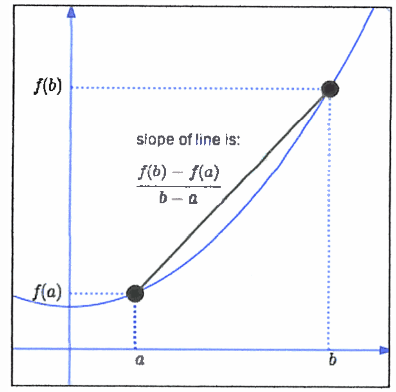
   $D = \{-1, 1, 3\}$

❖ AVERAGE RATE OF CHANGE

- An important application of average rate of change is comparing the change in the output quantity for a function as the input quantity changes.
- The average rate of change of a function f over an interval $[a, b]$ is calculated by dividing the difference of two outputs by the difference in the corresponding inputs. That is,

$$\text{average rate of change} = \frac{f(b) - f(a)}{b - a}$$

where $(a, f(a))$ & $(b, f(b))$ are any two data points in the function.



EXAMPLE:

5. Calculate the average rate of change on the interval $0 \le x \le 3$ of the functions $f(x)$, $g(x)$, & $h(x)$. Order the average rate of change from greatest to least.

$f(x)$

x	f(x)
0	1
1	2
2	5
3	7

AROC $= \frac{7-1}{3-0} = \frac{6}{3} = 2$

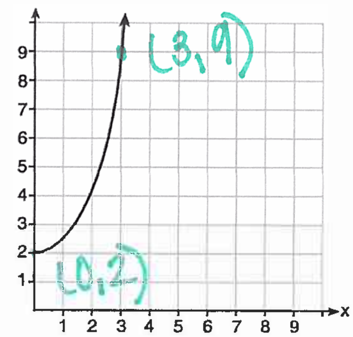
$g(x) = x^2 - 2x$

$g(0) = 0^2 - 2(0) = 0$
$g(3) = 3^2 - 2(3) = 3$

AROC $= \frac{3-0}{3-0} = \frac{3}{3} = 1$

Least

$h(x)$



AROC $= \frac{9-2}{3-0} = \frac{7}{3}$

Greatest

order: $h(x), f(x), g(x)$

1.3.D1 ~ Functions Represented by Tables

Objectives:

- Identify and interpret a functional relationship between two variables
- Evaluate functions and solve function equations for a given variable
- Determine the practical domain and range of a real-world situation

❖ EVALUATING FUNCTIONS

- The process of finding the *output* of a function that corresponds with a given *input*

❖ SOLVING FUNCTION EQUATIONS

- The process of finding the *input* of a function that corresponds with a given *output*

EXAMPLES: Use the table to fill in the missing values.

1. Evaluate: $f(-2) = 10$
2. Evaluate: $f(1) = -8$
3. Solve: $f(?) = 0$

$x = -3, 0, 3$ output

4. Solve: $f(?) = -10$

$x = 2$

x	y	
−3	0	
−2	10	$= f(-2)$
−1	8	
0	0	
1	−8	$= f(1)$
2	−10	
3	0	

❖ Determining if a Table Represents a Function

- To determine if a table represents a function, verify that each input value corresponds with a single output value.

EXAMPLE:

5. Consider the following table listing the official high temperature (in °F) in the village of Lake Placid, New York, during the first week of January. Note that the date has been designated the input and the high temperature on that date is the output.

Date (input)	1	2	3	4	5	6	7
Temperature (output)	25	30	32	24	23	27	30

a. Is the high temperature a function of the date? Explain your reasoning.

yes, each date has only one temperature

b. If the temperature is the input, is the date a function of the temperature? Explain.

NO, the input value of 30 has 2 different output values

❖ Predicting Unknown Data Values Using a Table

- Despite the limitations of using the average rate of change to predict unknown data values, we often use it in forecasting when additional data are unavailable.
- Using the average rate of change is frequently more accurate than guessing.

$$\text{average rate of change} = \frac{f(b) - f(a)}{b - a}$$

EXAMPLE:

6. *Perception reaction time distance* is the distance a vehicle will travel during the time when a hazard first becomes visible to a driver and the driver takes action to avoid the hazard.

Perception Time of 1.4 Seconds for Nominal Hazard	
Miles Per Hour m	Distance (in feet) D
30	61.60
40	82.13
50	102.67
60	123.20

Source: www.harristechnical.com/articles/skidmarks.pdf

a. What is the average rate of change of the distance over the interval $30 \leq m \leq 40$?

$$AROC = \frac{82.13 - 61.60}{40 - 30} = \frac{20.53}{10} = 2.053$$

b. Explain the meaning of the rate of change in the context of the data.

The output is increasing/decreasing by how much (include units) how often.

the distance is increasing by 2.053 feet for every mile per hour.

c. Use the average rate of change, calculated in part a, to estimate $f(36)$.

$$f(36) = f(30) + 6 \cdot AROC$$
$$= 61.60 + 6(2.053) = 73.918 \text{ feet}$$

d. How would you estimate the perception reaction time distance of a vehicle traveling 64 mph?

Find the AROC for the entire interval & calculate $f(60) + 4 \cdot AROC$

❖ PRACTICAL DOMAIN & RANGE

- In many real-world situations, not all values make sense for the domain and/or the range. For example, distance cannot be negative; number of people cannot be a decimal or a fraction.
  - In such situations, the values that make sense for the domain and range are called the practical (reasonable) domain and range.

EXAMPLE:

7. You probably need to fill your car with gas more often than you would like. You commute to college each day and to a part-time job each weekend. Your car gets good gas mileage, but the recent dramatic fluctuation in gas prices has wreaked havoc on your budget.

   a. Assume you need 12.6 gallons to fill up your car. The cost of a fill-up is dependent on one variable, the price per gallon. Complete the following table:

Price per gallon	2.00	2.50	3.00	3.50	4.00
Cost of fill-up	25.20	31.50	37.80	44.10	50.40

   b. Let p represent the price of a gallon of gasoline pumped (input) and c represent the cost of the fill-up (output). Write a function equation that expresses c in terms of p.

   $C = 12.6p$

   c. Use the equation to find p such that $C(p) = 35.91$ and interpret the result.

   $12.6p = 35.91$
   $p = 2.85$

   $C(2.85) = 35.91$

   price per gallon   cost of fill-up

   d. Can any number be substituted for the input variable in the cost-of-fill-up function? Describe the values that make sense and explain why they do.

   Negative values don't make sense & I doubt gas will ever be free.

   e. Determine the practical domain of the "cost of a fill-up" function.

   $0.01 \le p \le 4$

   f. What is the practical range of the "cost of a fill-up" function?

   $0.126 \le C \le 50.40$

Answers can vary.

1.3.D2 ~ Functions Represented by Formulas

Objectives:
- Evaluate functions and solve function equations for a given variable
- Create & use basic function formulas to model real-world situations
- Determine the practical domain and range of a real-world situation

❖ FUNCTIONS REPRESENTED BY FORMULAS

➢ A formula is a succinct mathematical statement expressing a relationship between quantities.
  - A formula is developed by generalizing the process of using input quantities to determine output quantities.

EXAMPLES:

1. The function $C = 3.5x + 2.8$ represents the cost of a taxi ride as a function of the distance (in miles) driven. You have enough money to travel at most 20 miles in the taxi. Determine the function's practical domain and range.

   D= $0 \le x \le 20$ miles   R= $2.80 \le C \le 72.80$ dollars

2. At the beginning of the semester, your buy a cafeteria meal ticket worth \$150. The daily lunch special costs \$4 in the college cafeteria.
   a. Fill in the following table.

Lunch specials purchased, n	0	10	20	30	35
Remaining balance, B	150	110	70	30	10

   b. Write a function formula that describes the relationship between the lunch specials purchased, n, and the remaining balance, B. $B = 150 - 4n$

   c. Determine the practical domain and range of the function.

   $150 - 4n = 0$
   $-4n = -150$
   $n = 37.5$

   D= $0 \le n \le 37$ lunch specials
   R= $2 \le B \le 150$

3. A student has a monthly car payment of \$172.55 a month for 6 years on a 2007 Toyota Yaris valued at \$13,210 in June 2006. Her down payment was \$3000.
   c. Write a function formula for the total cost of the car, T, as a function of the number of months she will have to pay on the loan, m. $T(m) = 3000 + 172.55m$

   d. Determine the practical domain and range of the "total cost" function.

   D= $0 \le m \le 72$ months   R= $3000 \le T \le 15423.60$ dollars

   e. How much has she paid in interest for her Toyota?

   $15423.60 - 13210 = \$2213.60$

1.4 ~ Functions Represented by Graphs

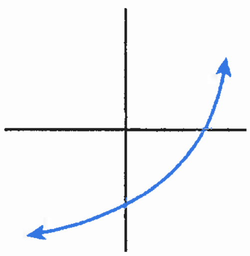
<u>Objectives:</u>

- Given the graph of a function: analyze, evaluate, and solve function equations for a given variable
- Describe (in words) what a graph tells you about a given situation
- Find a function's practical domain & practical range value from its graph
- Determine the vertical intercept (initial value) and horizontal intercepts of a function from a graph & interpret their real-world meanings

❖ Increasing, Decreasing, or Constant

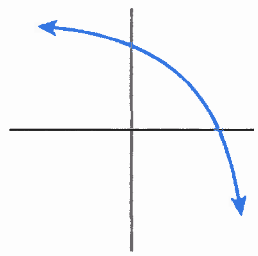
➢ Graphs are always constructed so that as you read the graph from left to right, the input variable increases in value. The graph shows change (increasing, decreasing, or constant) in the output values as the input values increase.

Increasing



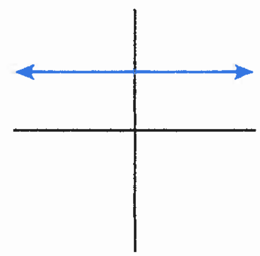
As the output value increases, the graph rises to the right.

Decreasing



As the output value decreases, the graph falls to the right.

Constant



If the output values are constant, the graph remains horizontal.

❖ Maximum & Minimum Points

➢ The point where a graph changes from rise to falling is called a local maximum point.

- The y-value of this point is called a local maximum value.

➢ The point where a graph changes from falling to rising is called a local minimum point.

- The y-value of this point is called a local minimum value.

❖ Evaluating Functions

➢ The process of finding the *output* of a function that corresponds with a given *input* is called evaluating a function.

- Is it possible for an "evaluate" problem to have more than one solution? Explain your reasoning.

No, in a function each input has exactly one output.

❖ Solving Function Equations

➢ The process of finding the *input* of a function that corresponds with a given *output* is called solving a function equation.

- Is it possible for a "solve" problem to have more than one solution? Explain your reasoning.

yes:

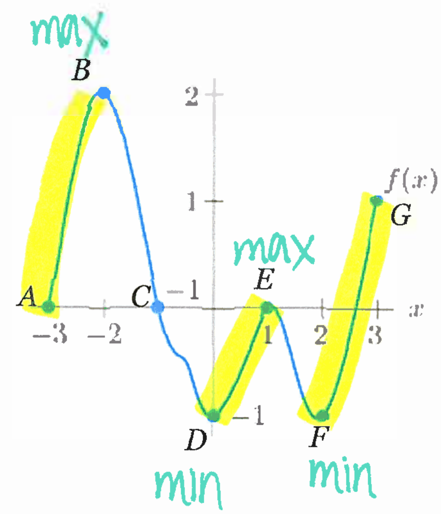
$x+1=0$
$x=-1$
one solution

$x^2-1=0$
$x^2=1$
$x=\pm 1$
two solutions

intervals (x start , x end)

EXAMPLES: Refer to the graph of $f(x)$:

1. For what interval(s) is the function increasing?
   $[-3,-2) \cup (0,1) \cup (2,3)$
2. For what interval(s) is the function decreasing?
   $(-2,0) \cup (1,2)$
3. Which point(s) indicate a local maximum? B & E
4. Which point(s) indicate a local minimum? D & F



5. Evaluate $f(-2)$ = 2
6. Evaluate $f(3)$ = 1
7. Solve $f(x) = 0$.
   $x = -3, -1, 1, 2.6$
8. Solve $f(x) = -1$.
   $x = 0$ & 2

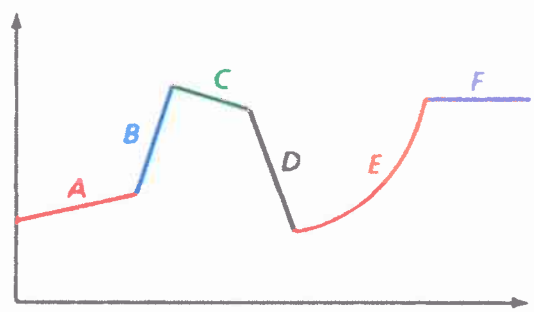
❖ **INTERPRETING GRAPHS IN CONTEXT**

➢ It is important to keep track of the independent and dependent variables, and to understand the contextual meaning of the coordinates of a point.
   - The graphs of functions intersect the vertical & horizontal axes at what we called the vertical & horizontal intercepts
     - Initial value: a vertical intercept

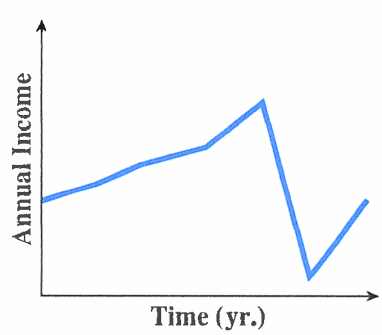
EXAMPLES:

Match the verbal description with the part of the graph it describes.

9. Stays the same F
10. Slowly decreases at a constant rate C
11. Slowly increases at a constant rate A
12. Increases at an increasing rate E
13. Quickly decreases at a constant rate D
14. Quickly increases at a constant rate B



15. The graph shows how the annual gross income changes in relation to the time (in years). Identify the input variable and the output variable. Then, interpret the situation; that is, describe, in words, what the graph is telling you about the situation.

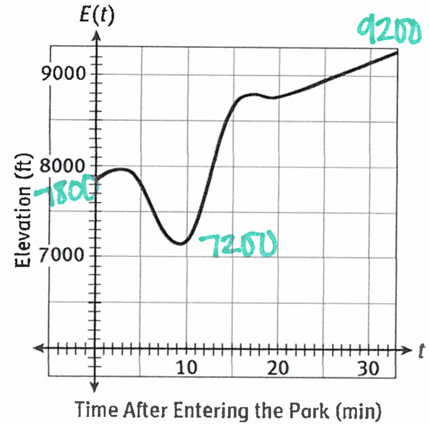


Input variable: time

Output variable: annual income

Interpretation: the annual income slowly increase, quickly decreases, & then begins to increase.

16. While on vacation, Jorge and Jackie traveled to Bryce Canyon National Park in Utah. They were impressed by the differing elevations at the viewpoints along the road. The graph describes the elevations for several viewpoints in terms of the time since they entered the park.



a. Does the graph represent a function? Explain.

yes, each input has only one output.

b. Identify the practical domain and range.

D= [0, 32] minutes

R= [7200, 9200] feet

c. What is the y-intercept? Interpret the meaning of the y-intercept in the context of the problem.

(0, 7800)

when Gage & Jackie entered the park their elevation was 7800 feet.

d. Identify an increasing interval.

(9, 18)

e. Identify a local minimum of the graph.

(9, 7200)