3.1 Exponential & Logistic Functions

Objective: Evaluate Exponential Expressions and Identify and Graph Exponential & Logistic Functions.

Exponential Functions

- Let a & b be real number constants. An exponential function in x is a function that…
  - Initial value of $f(x)$: the constant, a
  - Base: b

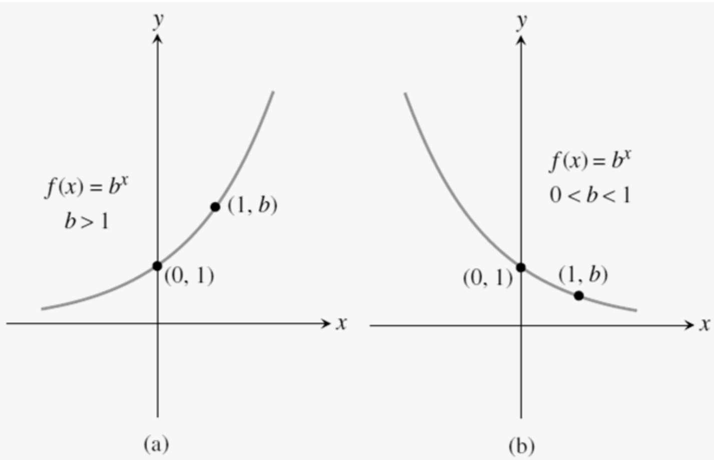


FIGURE 3.3 Graphs of $f(x) = b^x$ for (a) $b > 1$ and (b) $0 < b < 1$.

Domain: All reals
Range: $(0, \infty)$
Continuous
No symmetry: neither even nor odd
Bounded below, but not above
No local extrema
Horizontal asymptote: $y = 0$
No vertical asymptotes

If $b > 1$ (see Figure 3.3a), then
- f is an increasing function,
- $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) = \infty$.

If $0 < b < 1$ (see Figure 3.3b), then
- f is a decreasing function,
- $\lim_{x\to-\infty} f(x) = \infty$ and $\lim_{x\to\infty} f(x) = 0$.

Objective: Identify Exponential Functions

Examples: Identifying Exponential Functions
Which of the following are exponential functions? For those that are, state the initial value and the base. For those that are not, explain why.

1. $y = 5^x$

2. $y = x^{x^{1/2}}$

Objective: Evaluate Exponential Expressions

Computing Exponential Function Values

- When the inputs are rational numbers, use the ______________________________.

- We cannot use the properties of exponents to express an exponential function's value for irrational inputs.

Examples: **Computing Exponential Function Values (for Rational Numbers)**
Compute the exact value of the function for the given x-value without using a calculator.

3. $f(x) = 3 \cdot 5^x, x = 0$
4. $f(x) = -2 \cdot 3^x, x = {}^2/_3$

Objective: Write Exponential Functions

Example: **Finding an Exponential Function from its Table of Values**
Determine a formula for the exponential function whose values are given.

5.

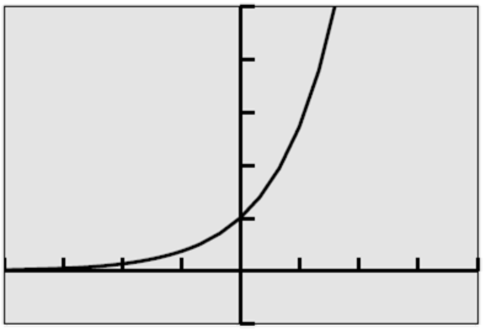
x	$f(x)$
−2	6
−1	3
0	3/2
1	3/4
2	3/8

Objective: Analyze & Graph Exponential & Logistic Functions

- Exponential Growth & Decay
  - For any exponential function $f(x) = a \cdot b^x$ & any real number x,
    $$f(x+1) = b \cdot f(x)$$
    - If $a > 0$ & $b > 1$, the function f is increasing & is an exponential growth function.
      - Base b: growth factor
    - If $a > 0$ & $b < 1$, f is decreasing & is an exponential decay function.
      - Base b: decay factor

- The Natural Exponential Function
  - The Natural Base e
    $$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$$
      - $e \approx 2.718281828459$

Basic Function: The Natural Exponential Function

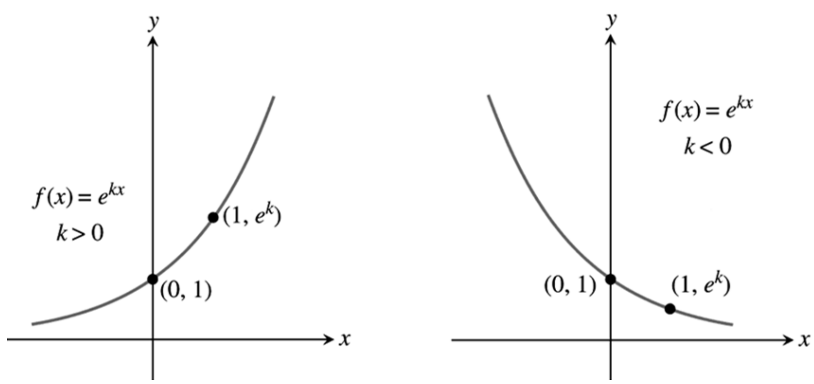


[–4, 4] by [–1, 5]

FIGURE 3.5 The graph of $f(x) = e^x$.

$f(x) = e^x$
Domain: All reals
Range: $(0, \infty)$
Continuous
Increasing for all x
No symmetry
Bounded below, but not above
No local extrema
Horizontal asymptote: $y = 0$
No vertical asymptotes
End behavior: $\lim_{x \to -\infty} e^x = 0$ and $\lim_{x \to \infty} e^x = \infty$

- Exponential Functions & the Base e
  - Any exponential function $f(x) = a \cdot b^x$ can be rewritten as: $f(x) = a \cdot e^{kx}$ for an appropriately chosen real number constant k.
    - If $a > 0$ & $k > 0$, $f(x) = a \cdot e^{kx}$ is an exponential growth function.
    - If $a < 0$ & $k < 0$, $f(x) = a \cdot e^{kx}$ is an exponential decay function.



Examples: **Exponential Growth or Decay?**

State whether the function is an exponential growth function or exponential decay function & describe its end behavior using limits.

6. $f(x) = 3^{-2x}$

7. $f(x) = 0.5^x$

Logistic Functions & Their Graphs

- Exponential vs. Logistic

<u>Exponential</u>

- Unrestricted
- Increases at an ever increasing rate
- Not bounded above

<u>Logistic</u>

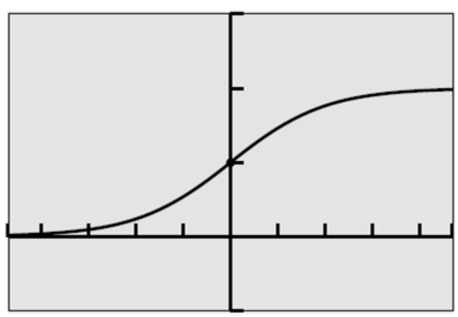
- Restricted
- Increases, eventually levels out
- Bounded above & below

- Logistic Growth Functions
  - Let a, b, c, and k be positive constraints, with b < 1. A logistic growth function in x is a function that can written in the form:

$$f(x) = \frac{c}{1 + a \cdot b^x} \ OR \ f(x) = \frac{c}{1 + a \cdot e^{-kx}}$$

where the constant c is the limit to growth.

BASIC FUNCTION: THE LOGISTIC FUNCTION



[–4.7, 4.7] by [–0.5, 1.5]

FIGURE 3.8 The graph of $f(x) = 1/(1 + e^{-x})$.

$f(x) = \dfrac{1}{1 + e^{-x}}$

Domain: All reals
Range: (0, 1)
Continuous
Increasing for all x
Symmetric about (0, 1/2), but neither even nor odd
Bounded below and above
No local extrema
Horizontal asymptotes: $y = 0$ and $y = 1$
No vertical asymptotes
End behavior: $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to \infty} f(x) = 1$

- More on the Graphs of Logistic Functions
  - End behavior is always described by the equations:

$$\lim_{x \to -\infty} f(x) = 0^+ \ \& \ \lim_{x \to \infty} f(x) = c^-$$

c is the limit to growth

  - Bounded by their horizontal asymptotes: $y = 0$ & $y = c$
  - Range: $(0, c)$
  - Symmetric about the point of its graph with y-coordinate, $\frac{c}{2}$
    - Usually <u>not</u> the y-intercept

Example 8: **Graphing Logistic Growth Functions**

Graph the function using a grapher. Find the y-intercept and the horizontal asymptotes. Is the graph symmetric about its y-intercept?

$$f(x) = \frac{16}{1 + 3e^{-2x}}$$

3.2 Exponential & Logistic Function Models

Objective: Analyze & Write Exponential & Logistic Growth and Decay Function Models

- Constant Percentage Rate & Exponential Functions
  - Exponential Population Model
    - If a population P is changing at a constant percentage rate r each year, then…

      - P_0 – initial population
      - r – constant percentage rate (expressed as a decimal)
      - t – time (in years)

Exponential Growth Function

- $r > 0$
- Growth factor: $1 + r$

Exponential Decay Function

- $r < 0$ & the base $1 + r < 1$
- Decay factor: $1 + r$

Objective: Analyze Exponential Growth and Decay Models

Examples: **Finding Growth & Decay Rates**

Tell whether the function is an exponential growth or decay function & find the constant percentage rate of growth or decay.

1. $P(t) = 3.5 \cdot 1.09^t$

2. $f(x) = 78{,}963 \cdot 0.968^x$

3. $g(t) = 247 \cdot 2^t$

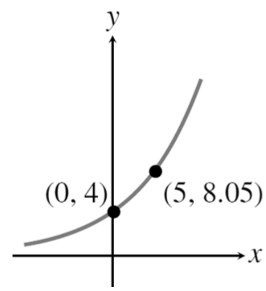
Objective: Write Exponential Functions

Examples: **Finding an Exponential Function**
Determine the exponential function that satisfies the given conditions.

4. Initial population = 28,900, decreasing at a rate of 2.6% per year

5. Initial value = 5, increasing at a rate of 17% per year

6. Determine a formula for the exponential function whose graph is shown:

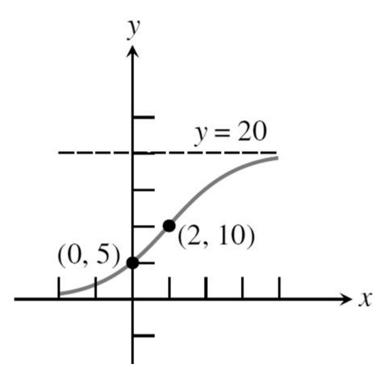


Objective: Write Logistic Functions

Examples: **Finding a Logistic Function**
Find the logistic function that satisfies the given conditions.

7. Initial value = 10, limit to growth = 40, passing thru (1, 20)

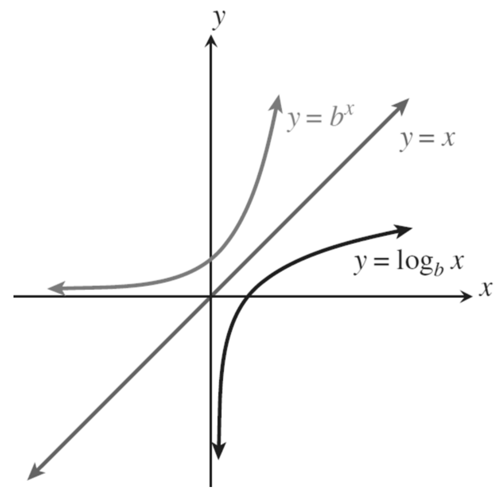
8. Determine a formula for the logistic function whose graph is shown in the figure.



3.3 Logarithmic Functions & Their Graphs

Objective: Convert Equations between Logarithmic Form and Exponential Form, Evaluate Common and Natural Logarithms, and Graph Common and Logarithmic Functions

- Inverses of Exponential Functions
  - Logarithmic Function with Base b
    $If\ f(x) = b^x\ with\ b > 0\ \&\ b \neq 1, then\ f^{-1}(x) = \log_b x.$



Objective: Convert Equations between Logarithmic Form and Exponential Form

- Changing Between Logarithmic & Exponential Form
  - A logarithm is an exponent
    - Use the properties of exponents to evaluate logarithms
- Basic Properties of Logarithms
  - For $0 < b \neq 1, x > 0$, and any real number y,
    - $\log_b 1 = 0\ because\ b^0 = 1$
    - $\log_b b = 1\ because\ b^1 = b$
    - $\log_b b^y = y\ because\ b^y = b^y$
    - $b^{\log_b x} = x\ because\ \log_b x = \log_b x$

Recall that:

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

$$\frac{1}{a^n} = a^{-n}$$

Examples: **Evaluating Logarithmic & Exponential Expressions**
Use the "Basic Properties of Logarithms" to evaluate each expression.

1. $\log_3 27$
2. $\log_2 \sqrt{2}$
3. $\log_4 \frac{1}{16}$
4. $\log_5 1$

Objective: Evaluate Common and Natural Logarithms

Common Logarithms – Base 10

- We often drop the subscript of 10 for the base when using common logarithms.

➢ The common logarithmic function $\log_{10} x = \log x$ is the inverse of the exponential function $f(x) = 10^x$.

➢ Basic Properties of Common Logarithms

- Let x & y be real numbers with $x > 0$,
  - $\log 1 = 0$ *because* $10^0 = 1$
  - $\log 10 = 1$ *because* $10^1 = 10$
  - $\log 10^y = y$ *because* $10^y = 10^y$
  - $10^{\log x} = x$ *because* $\log x = \log x$

Recall that:

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

$$\frac{1}{a^n} = a^{-n}$$

Examples: **Evaluating Logarithmic & Exponential Expressions – Base 10**
Evaluate each expression without using a calculator.

5. $\log \sqrt[4]{10}$

6. $10^{\log 7}$

7. $\log 100{,}000$

8. $\log \dfrac{1}{10{,}000}$

Solving Simple Logarithmic Equations

➢ Changing from logarithmic form to exponential form sometimes is enough to solve an equation involving logarithmic functions.

Examples: **Solving Simple Logarithmic Equations**

9. $\log x = 2$

10. $\log x = -1$

Natural Logarithms – Base *e*

- Logarithms with base *e* are natural logarithms – "ln"

➢ The natural logarithmic function $\log_e x = \ln x$ is the inverse of the exponential function $f(x) = e^x$.

- Basic Properties of Natural Logarithms
  - Let x & y be real numbers with $x > 0$,
    - $\ln 1 = 0$ *because* $e^0 = 1$
    - $\ln e = 1$ *because* $e^1 = e$
    - $\ln e^y = y$ *because* $e^y = e^y$
    - $e^{\ln x} = x$ *because* $\ln x = \ln x$

Recall that:

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

$$\frac{1}{a^n} = a^{-n}$$

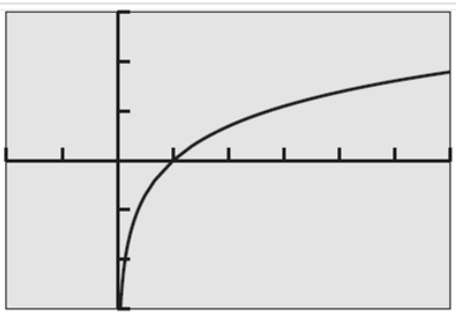
<u>Examples: **Evaluating Logarithmic & Exponential Expression – Base *e***</u>
Evaluate each expression without using a calculator.

11. $\ln \sqrt[5]{e}$ 12. $\ln e^{11}$ 13. $e^{\ln e}$

<u>Objective:</u> Analyze & Graph Common and Logarithmic Functions

Graphs of Logarithmic Functions

Basic Function: The Natural Logarithmic Function



[–2, 6] by [–3, 3]

$f(x) = \ln x$
Domain: $(0,\infty)$
Range: All reals
Continuous on $(0,\infty)$
Increasing on $(0,\infty)$
No symmetry
Not bounded above or below
No local extrema
No horizontal asymptotes
Vertical asymptote: $x = 0$
End behavior: $\lim_{x\to\infty} \ln x = \infty$

3.4 Properties of Logarithmic Functions

<u>Objective:</u> Apply the Properties of Logarithms to Evaluate Expressions and Graph Functions

Properties of Logarithms

- Let b, R & S be positive real numbers with $b \neq 1$, & c is any real number
  - Product rule
  - Quotient rule
  - Power rule

- Expanding or Condensing Logarithms
  - Why is this necessary?
    - When solving equations involving logarithms, it may be necessary to expand (or condense) logarithms.

Objective: Apply the Properties of Logarithms

Examples: **Expanding Logarithms**

Assuming x & y are positive, use properties of logarithms to write each expression as a sum of logarithms or multiples of logarithms.

1. $\log(25xy^3)$
2. $\ln\frac{\sqrt{x+1}}{x^2}$

Examples: **Condensing Logarithms**

Assuming x & y are positive, use properties of logarithms to write each expression as a single logarithm.

3. $\ln x^4 - 3\ln(x^2y)$

4. $3\ln x^2 + 2\ln(xy) - \ln(x^3y)$

- Change-of-Base Formula for Logarithms
  - For positive real numbers a, b & x with $a \neq 1$ & $b \neq 1$:

Examples: **Evaluate by Changing the Base**

Use the change-of-base formula & evaluate each logarithm. Round to four decimal places.

5. $\log_8 175$
6. $\log_{0.5} 12$

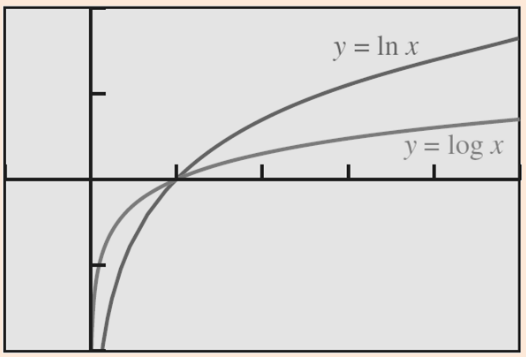
Objective: Use Properties of Logarithms to Graph Functions

- Graphs of Logarithmic Functions w/Base b
  - Any logarithmic function can be rewritten so that it is a constant multiple of the natural logarithmic function $f(x) = \ln x$

$$g(x) = \log_b x =$$

  - If $b > 1$, then there's a vertical stretch (or compression)
  - If $0 < b < 1$, then there's a reflection across the x-axis as well

Logarithmic Functions: $f(x) = \log_b x$, with $b > 1$



[–1, 5] by [–2, 2]

Domain: $(0, \infty)$
Range: All reals
Continuous
Increasing on its domain
No symmetry; neither even nor odd
Not bounded above or below
No local extrema
No horizontal asymptotes
Vertical asymptote: $x = 0$
End behavior: $\lim_{x\to\infty} x = \infty$

Examples: **Graphing Logarithmic Functions**

Describe how to transform the graph of $f(x) = \ln x$ into the graph of the given function. Verify by graphing the function w/a grapher.

7. $g(x) = \log_3 x$

8. $h(x) = \log_{1/3} x$

3.5 Equation Solving

Objective: Apply the Properties of Logarithms to Solve Exponential and Logarithmic Equations Algebraically

- Solving Exponential Equations
  - One-to-One Properties
    - For any exponential function: $f(x) = b^x$
      -

- For any logarithmic function: $f(x) = \log_b x$
  -

Examples: **Solving an Exponential Equation Algebraically**

Find the exact solution algebraically, and check it via substitution.

1. $81\left(\frac{2}{3}\right)^{x/4} = 16$
2. $2\left(10^{-x/3}\right) = 20$

Solve the equation – for x – algebraically. Calculate a numerical approximation and check it via substitution.

3. $50e^{0.035x} = 200$
4. $1.06^x = 4.1$

Solve graphically; round your solution to the nearest thousandth.

5. $\dfrac{2^x - 2^{-x}}{3} = 4$

- Solving Logarithmic Equations
  - Methods
    - Use the one-to-one property of logarithms
    - Change the equation from logarithmic to exponential form
    - Use the power rule of logarithms
    - Graph each side of the equation separately
    - Set the equation equal to 0, graph & find x-intercept(s)

- IMPORTANT!
  - Keep track of the domain of each expression in the equation as it is being solved.
    - A particular algebraic method may introduce extraneous solutions or worse yet lose some valid solutions.

Examples: **Solving a Logarithmic Equation**

Find the exact solution algebraically, and check it via substitution.

6. $\log_4(x-5) = -1$
7. $\log x^2 = 8$

Solve the equation algebraically. Obtain a numerical approximation and check it via substitution.

8. $3\ln(x-3) + 4 = 5$
9. $\ln(x-3) + \ln(x+4) = 3\ln 2$