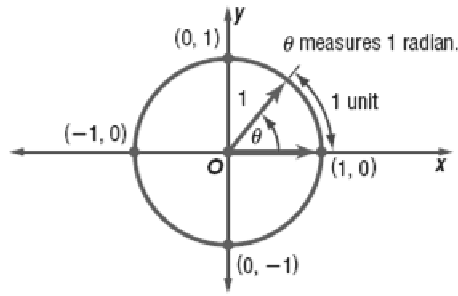
4.1 Angles & Their Measure

Objectives:

- Degrees & Radians
  - Degrees – a unit of angular measure equal to $^1/_{180}$th of a straight angle
    - D°M'S" - 1° = 60 minutes = 3600 seconds
  - Radians
    - A central angle of a circle has measure 1 radian if it intercepts an arc w/the same length as the radius.
  - Degree-Radian Conversion
    - Radians → degrees:
    - Degrees → radians:



Examples: Working with DMS & Radian Measure

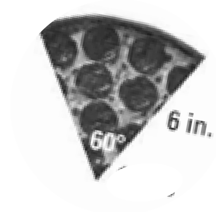
1. Convert 23.325° to DMS.
2. Convert 30°30'18" to degrees.
3. How many radians are in 55°?
4. How many degrees are in $\frac{\pi}{10}$ radians?

- Circular Arc Length
  - Arc Length
    - If θ is a central angle (in radians) in a circle of radius r, and if θ is measured in radians, then the length s of the intercepted arc is:

Examples: Circular Arc Length

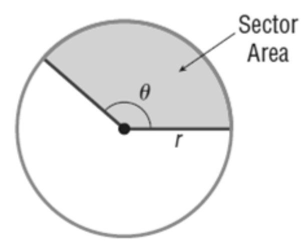
5. Find the length of an arc intercepted by a central angle of $^1/_8$ radian in a circle of radius 24 inches.

6. Find the perimeter of a 60° slice of a small (6 in. radius) pizza.



- Area of a Sector
  - A sector is a region of a circle that is bounded by a central angle θ and its intercepted arc.
    - The area A of a sector w/radius r and central angle θ is given by:

      where θ is measured in radians.



Examples: Area of a Sector

7. Find the area of a sector w/a central angle of $^{4\pi}/_{3}$ radians in a circle whose radius is 10 inches.

- Angular & Linear Motion
  - Linear Speed
    - Suppose that an object moves around a circle of radius r at a constant speed. If s is the distance traveled (AKA arc length) in time t around this circle, then linear speed v of the object is:
      - Measured in length/(unit of time)
  - Angular Speed
    - As an object travels around a circle, suppose that θ (measured in radians) is the central angle swept out in time t. Then the angular speed ω of this object is the angle swept out divided by the elapsed time:
      - Measured in units involving revolutions or radians
      - What is one revolution equivalent to?

- Circular Motion:

$$s = r\theta$$

$$\frac{s}{t} = \frac{r\theta}{t}$$ Divide both sides by t

$$\boldsymbol{v = r\omega}$$ Make these substitutions:

$$v = \frac{s}{t} \ \& \ \omega = \frac{\theta}{t}$$

<u>Examples: Using Angular Speed</u>

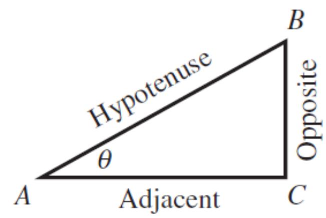
8. Clark's truck has wheels 30 inches in diameter. If the wheels are rotating at 600 rpm (revolutions per minute), find the truck's speed in miles per hour.

9. Kali Nguyen races on a bicycle w/13-inch radius wheels. When she is traveling at a speed of 44 ft/sec, how many revolutions per minute are her wheels making?

4.2 Trig Functions of Acute Angles

Objective:

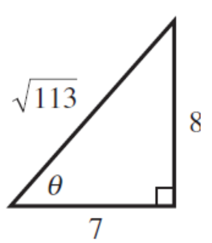
- Right Triangle Trig
  - A single acute angle θ of a right triangle determines six distinct ratios of side lengths.
    - An acute angle θ is in standard position, with one ray along the positive x-axis & the other extending in the first quadrant.
  - The Six Trig Functions
    - Let θ be an acute angle in right $\triangle ABC$, then…



Sine	$= \sin\theta$	$= \dfrac{opp}{hyp}$	Cosecant	$= \csc\theta$	$= \dfrac{hyp}{opp}$
Cosine	$= \cos\theta$	$= \dfrac{adj}{hyp}$	Secant	$= \sec\theta$	$= \dfrac{hyp}{adj}$
Tangent	$= \tan\theta$	$= \dfrac{opp}{adj}$	Cotangent	$= \cot\theta$	$= \dfrac{adj}{opp}$

Example 6: Evaluating Trig Functions

Find the values of all six trig functions of the angle θ.



$\sin\theta =$ ______ $\qquad \csc\theta =$ ______

$\cos\theta =$ ______ $\qquad \sec\theta =$ ______

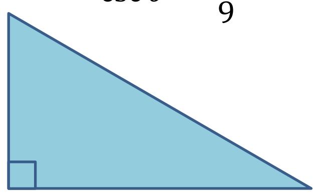
$\tan\theta =$ ______ $\qquad \cot\theta =$ ______

- Using One Trig Ratio to Find the Remaining Five
  - Given: θ as an acute angle & the ratio of one of the six trig functions
    - Sketch a reference triangle & label the sides given in the trig ratio
    - Use the Pythagorean Theorem to find the missing side
    - Use the definitions to find the remaining five trig function values

Example 7: Evaluating Trig Functions

Assume that θ is an acute angle in a right triangle satisfying the given condition. Evaluate the remaining trig functions:

$\csc\theta = \frac{23}{9}$



$\sin\theta =$ ______ $\csc\theta =$ ______

$\cos\theta =$ ______ $\sec\theta =$ ______

$\tan\theta =$ ______ $\cot\theta =$ ______

- Applications of Right Triangle Trig
  - Solving a Triangle – Using some of the parts of a triangle to solve for all the others

Example 8: Solving a Right Triangle

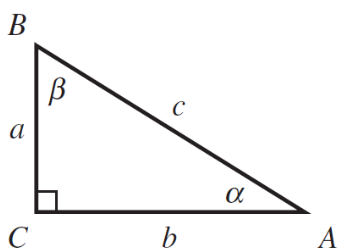
Solve the right triangle △ABC for all of its unknown parts

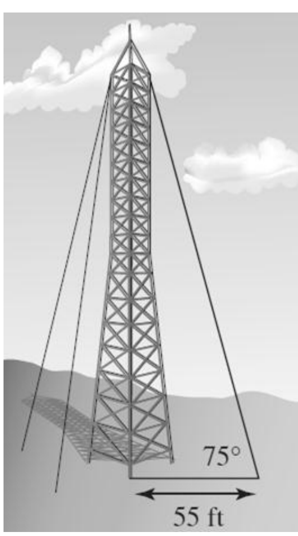
Given: $\beta = 55°$; $a = 15.58$

$\alpha =$ ________

$b =$ ________

$c =$ ________





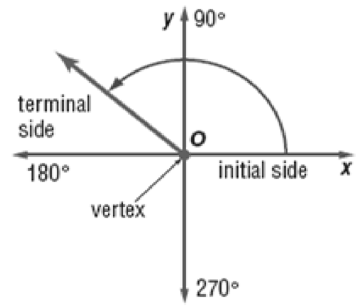
Example 9: Applications of Right Triangles

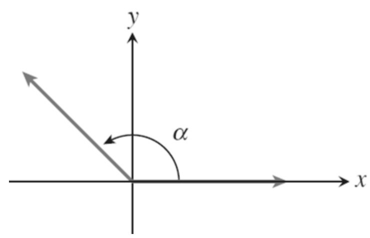
A guy wire from the top of the transmission tower at WJBC forms a 75° angle with the ground at a 55-foot distance from the base of the tower. How tall is the tower? (Approximate your solution to two decimal places.)

4.3 Trigonometry Extended: The Circular Functions

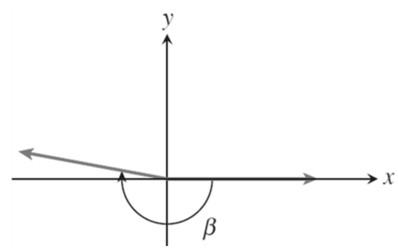
Objective:

- Trig Functions of Any Angle
  - In trigonometry, an angle is a rotating ray.
    - The beginning position of the ray – the <u>initial side</u> – is rotated about its endpoint – the <u>vertex</u> – to its final position – the <u>terminal side</u>.
    - An angle's measure = the amount of rotation
    - The angles below are in <u>standard position</u>; their initial side lies on the x-axis





A positive angle (counterclockwise)



A negative angle (clockwise)

- Coterminal Angles
  - Angles w/the same initial & terminal sides
    - Angles are coterminal whenever they differ by an integer multiple of 360° or an integer multiple of 2π

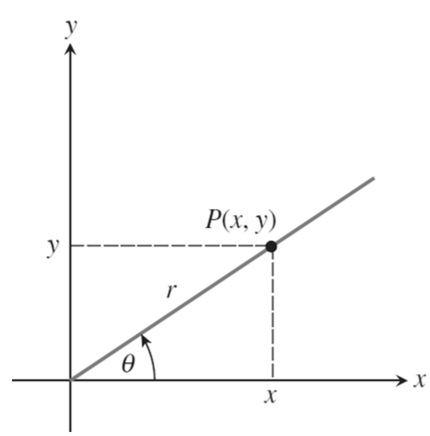
<u>Examples: Finding Coterminal Angles</u>

Find a positive & a negative angle that are coterminal w/the given angle.

1. $\theta = 45°$

2. $\theta = \dfrac{4\pi}{3}$

- First Quadrant Trigonometry
  - A point $P(x, y)$ in Quadrant I determines an acute angle θ.
  - The number r denotes the distance from P to the origin.



Example 3: Evaluating Trig Functions in Q1

Let θ be the acute angle in standard position whose terminal side contains the point (7, 3). Find the six trig functions of θ.

$\sin\theta =$ ______	$\csc\theta =$ ______
$\cos\theta =$ ______	$\sec\theta =$ ______
$\tan\theta =$ ______	$\cot\theta =$ ______

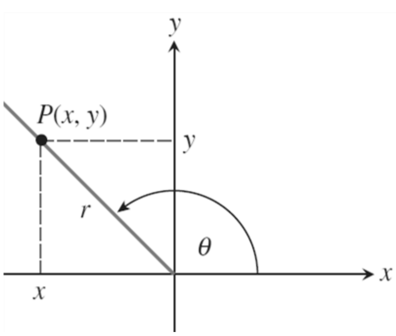
Example 4: Evaluating Trig Functions Not in Q1

Let θ be the acute angle whose terminal side contains the point (-2, -5). Find the six trig functions of θ.

$\sin\theta =$ ______	$\csc\theta =$ ______
$\cos\theta =$ ______	$\sec\theta =$ ______
$\tan\theta =$ ______	$\cot\theta =$ ______

Trigonometric Functions of Any Angle

Let θ be any angle in standard position and let $P(x, y)$ be any point on the terminal side of the angle (except the origin). Let r denote the distance from $P(x, y)$ to the origin, i.e.,

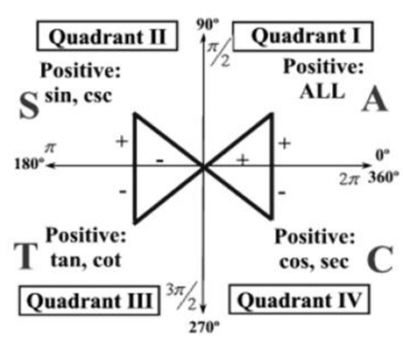


$$r = \sqrt{x^2 + y^2}$$

$\sin\theta = \dfrac{y}{r}$	$\cos\theta = \dfrac{x}{r}$	$\tan\theta = \dfrac{y}{x}$
$\csc\theta = \dfrac{r}{y}$	$\sec\theta = \dfrac{r}{x}$	$\cot\theta = \dfrac{x}{y}$

What do we do if we start with an angle θ in standard position and we want to evaluate the trigonometric functions?

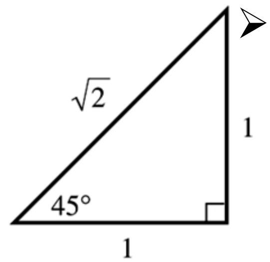
We try to find a point (x, y) on its terminal side.



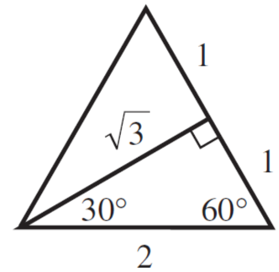
Remember to…
watch your SIGNS!

Two Famous Triangles

- $45° - 45° - 90°$



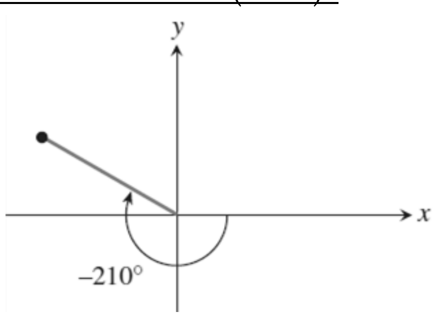
- $30° - 60° - 90°$

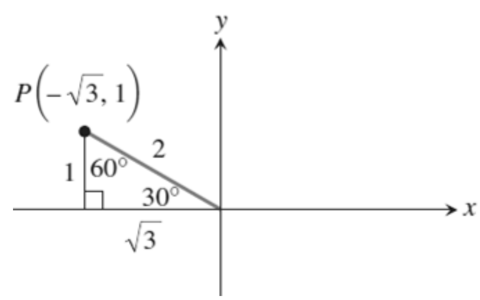


  - Exact Values of Special Right Triangles

	$\theta = 45°$	$\theta = 30°$	$\theta = 60°$
$\sin\theta$			
$\cos\theta$			
$\tan\theta$			
$\csc\theta$			
$\sec\theta$			
$\cot\theta$			

Example 5: What is sin(-210°)?





Reference Triangles

- For an angle θ in standard position, the reference triangle is the triangle formed by the terminal side of angle θ, the x-axis, and a perpendicular dropped from a point on the terminal side to the x-axis.

Reference Angles

- The angle in the reference triangle at the origin is the <u>reference angle</u>.
  - The reference angle will either be 30°, 45° or 60°
- Finding the reference angle:
  - Determine the quadrant in which the terminal side lies
    - Quad 1: $\theta^R = \theta$
    - Quad 2: $\theta^R = 180° - \theta$ OR $\theta^R = \pi - \theta$
    - Quad 3: $\theta^R = \theta + 180°$ OR $\theta^R = \theta + \pi$
    - Quad 4: $\theta^R = 360° - \theta$ OR $\theta^R = 2\pi - \theta$

<u>Examples: Evaluating More Trig Functions (continued)</u>

Find the six trig functions of the given angle (without a calculator). Draw a reference triangle in the proper quadrant. (Watch your <u>signs</u>!)

6. 240°

$\sin\theta =$ ______ $\csc\theta =$ ______

$\cos\theta =$ ______ $\sec\theta =$ ______

$\tan\theta =$ ______ $\cot\theta =$ ______

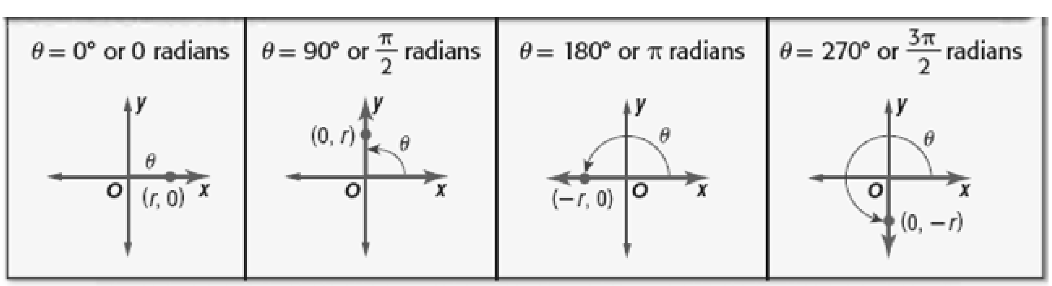
7. $\dfrac{3\pi}{4}$

$\sin\theta =$ ______ $\csc\theta =$ ______

$\cos\theta =$ ______ $\sec\theta =$ ______

$\tan\theta =$ ______ $\cot\theta =$ ______

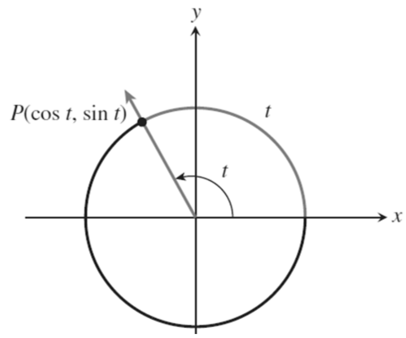
- <u>Quadrantal angles</u> – An angle whose terminal side lies along one of the coordinate axes



Complete the table…Watch your signs!

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
0° 0 radians						
90° $^{\pi}/_{2}$ radians						
180° π radians						
270° $^{3\pi}/_{2}$ radians						

- Circular Functions
  - When trig functions are applied to real numbers
    - The real number t determines a point on the unit circle
    - The (x, y) coordinates determine the six trig ratios



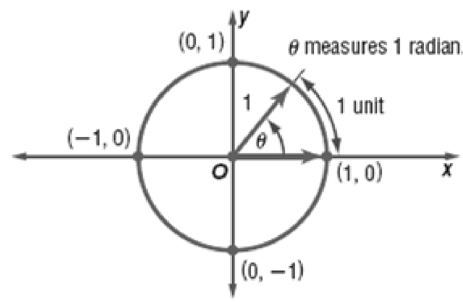
Example 8: Using One Trig Ratio to Find Another

Find the six trig functions of the given angle (without a calculator). Draw a reference triangle in the proper quadrant. (Watch your signs!)

$$\sin\theta = -\frac{2}{5} \;\&\; \cos\theta > 0$$

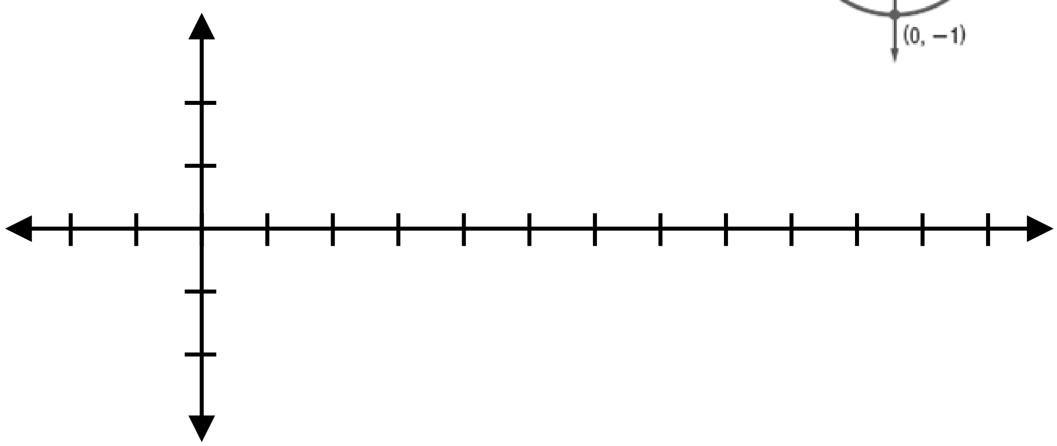
4.4 Graphs of Sine & Cosine: Sinusoids

Objectives:

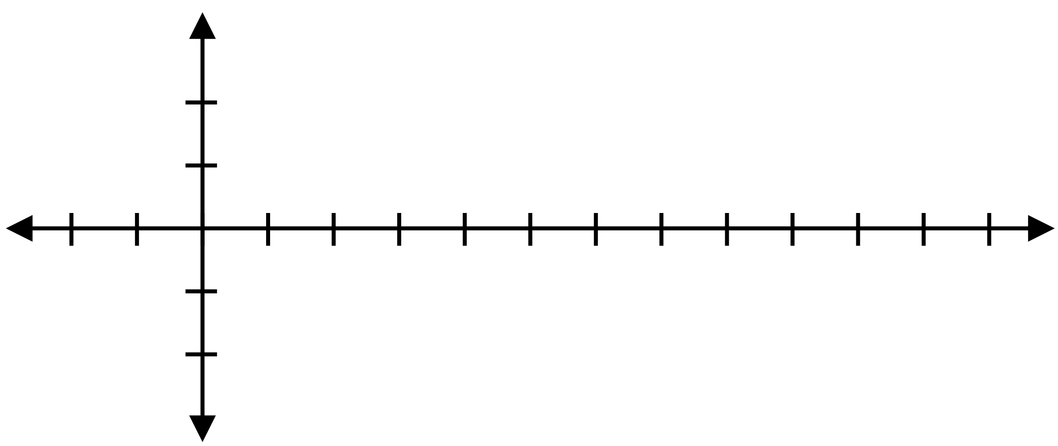


Use the unit circle to derive the graphs of sine and cosine.

- The Sine Function: $f(x) = \sin x$



- The Cosine Function: $f(x) = \cos x$



	Period	Domain	Range	Zeros	Even/Odd
Sine					
Cosine					

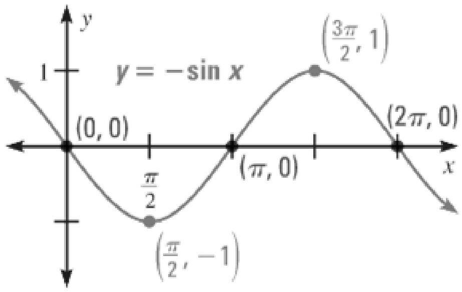
- Sinusoids
  - What transformation can be performed on the sine function to become the cosine function?

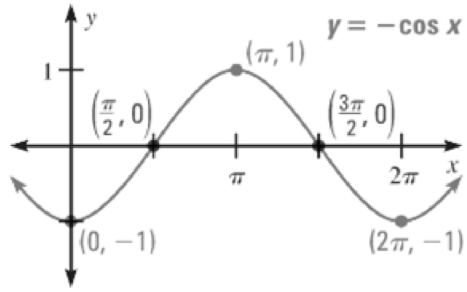
  - A function is a <u>sinusoid</u> if it can be written in the form:

$$\boldsymbol{f(x) = a \sin b(x - c) + d}$$

    where a, b, c & d are constants & neither a nor b is 0.
    - Cosine is a sinusoid.

- Sinusoids & Transformations
  - Amplitude - $|\ a\ |$
    - Half the height of the wave
      - Graphing transformation: vertical stretch, if $|\ a\ | > 1$, or compression, if $|\ a\ | < 1$
  - Period – b
    - The length of one full cycle of the wave
    - The period of sine & cosine is 2π
      - Graphing transformation: horizontal stretch if $|\ b\ | < 1$, or compression if $|\ b\ | > 1$ (both by a factor of $1/_{|\ b\ |}$)
      - If $b < 0$, there's a reflection across the y-axis
    - To determine the "graphing period" divide 2π by $|\ b\ |$.
  - Frequency – $1/_{\mathrm{b}}$
    - The number of complete cycles the wave completes in an interval
      - $|\ \mathrm{b}\ |/_{2\pi}$
  - Phase shift – c
      - Graphing transformation: right: $x - c$ or left: $x + c = x - (-c)$
  - Vertical shift – d
      - Graphing transformation: up: $+\ d$ or down: $-\ d$
  - Caution!
    - A change in the period also affects the phase shift!
    - Reflections cause all sorts of problems.
      - $y = -\sin x$ or $y = -\cos x$

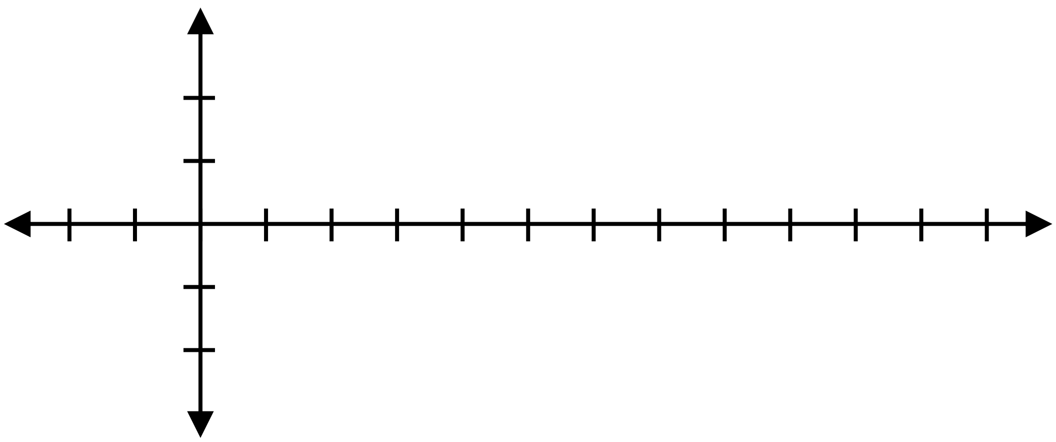




Examples: Sine & Cosine Functions

1. Find the amplitude, period and frequency of the function. Use this information to sketch a graph of the function:

   $$y = -2\cos\frac{x}{2}$$



2. Identify the maximum and minimum values and the zeros of the function in the interval [-2π, 2π]. (Use your understanding of transformations and not your graphers.)

   $$y = \frac{1}{2}\sin 2x$$

3. Describe the transformations required to obtain the graph of the given function from the basic cosine function:

   $$y = 3\cos\frac{2\pi x}{3}$$

4. Construct the sinusoid with an amplitude of 1.5 and a period of $\pi/6$ that goes through (1, 0).

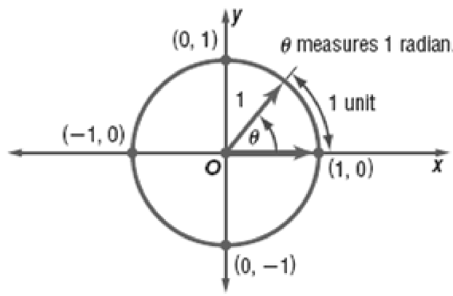
5. State the amplitude and period of the sinusoid, and (relative to the basic function) the phase shift and vertical translation.

	$(a)\ y = -2\sin\left(x - \frac{\pi}{4}\right) + 1$	$(b)\ y = 5\cos\left(3x - \frac{\pi}{6}\right) + 0.5$
Amplitude		
Period		
Phase shift		
Vertical shift		
Other transformations?		

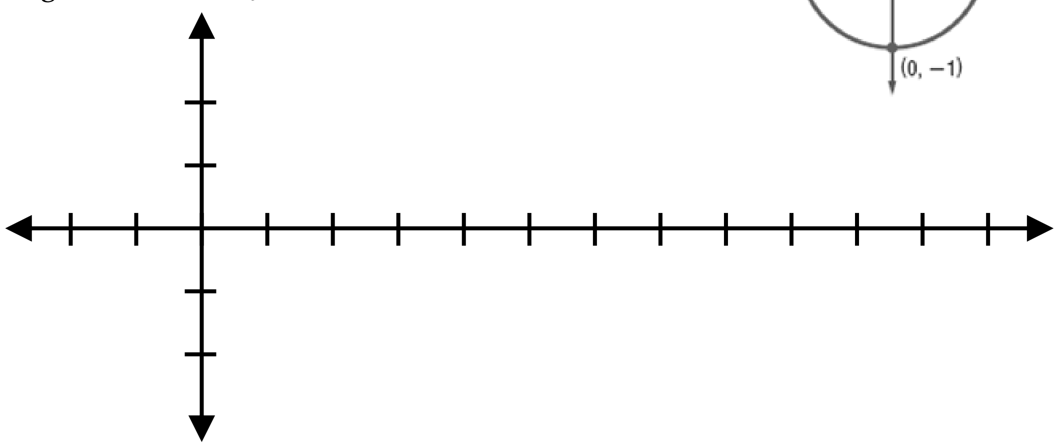
4.5 Graphs of Tangent, Cotangent, Secant & Cosecant

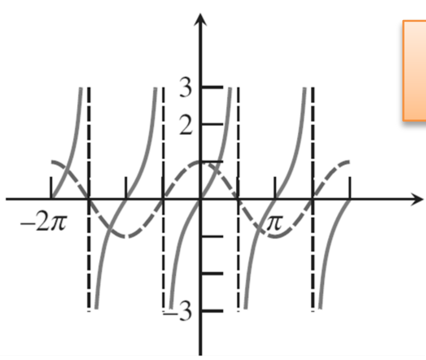
Objectives:

Use the unit circle to derive the graphs of tangent and cotangent.



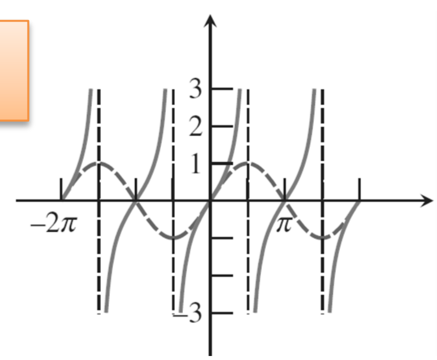
ට The Tangent Function: $f(x) = \tan x$





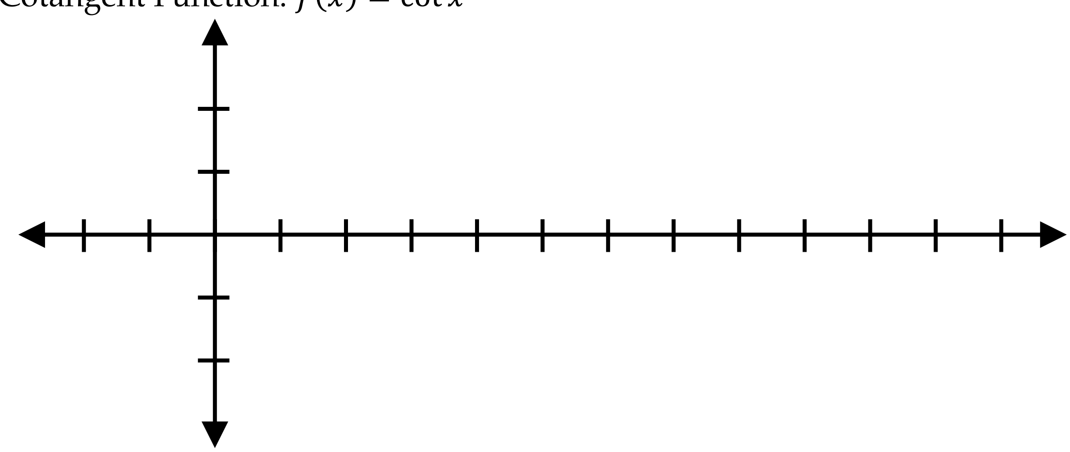
The tangent function has asymptotes at the zeros of cosine.

$$\tan x = \frac{\sin x}{\cos x}$$

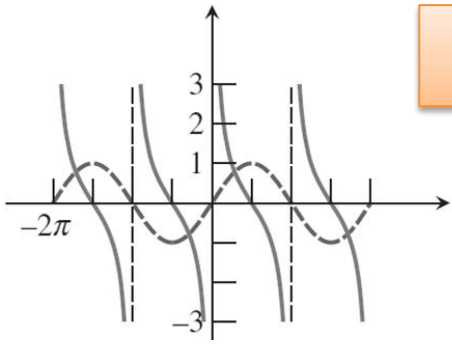


The tangent function has zeros at the zeros of sine.

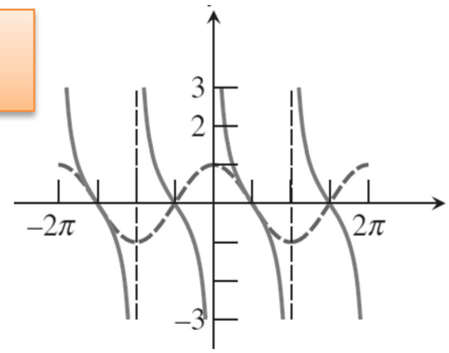
ට The Cotangent Function: $f(x) = \cot x$



$$\cot x = \frac{\cos x}{\sin x}$$



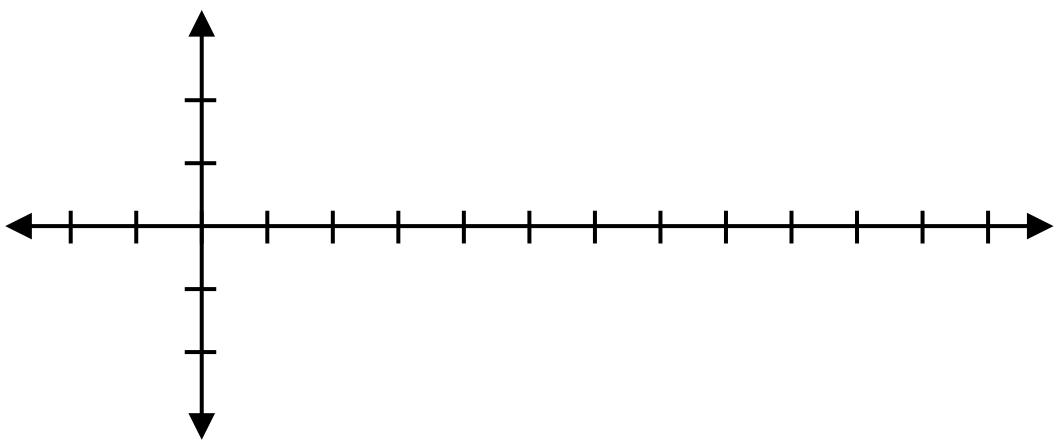
The cotangent function has asymptotes at the zeros of sine.



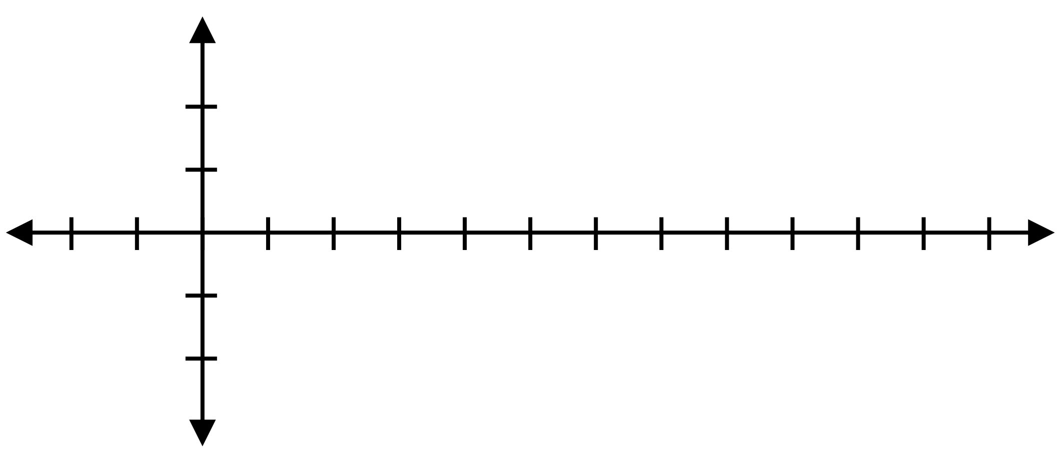
The cotangent function has zeros at the zeros of cosine.

Use the graphs of sine and cosine to derive the graphs of secant and cosecant.

- The Cosecant Function: $f(x) = \csc x$



- The Secant Function: $f(x) = \sec x$

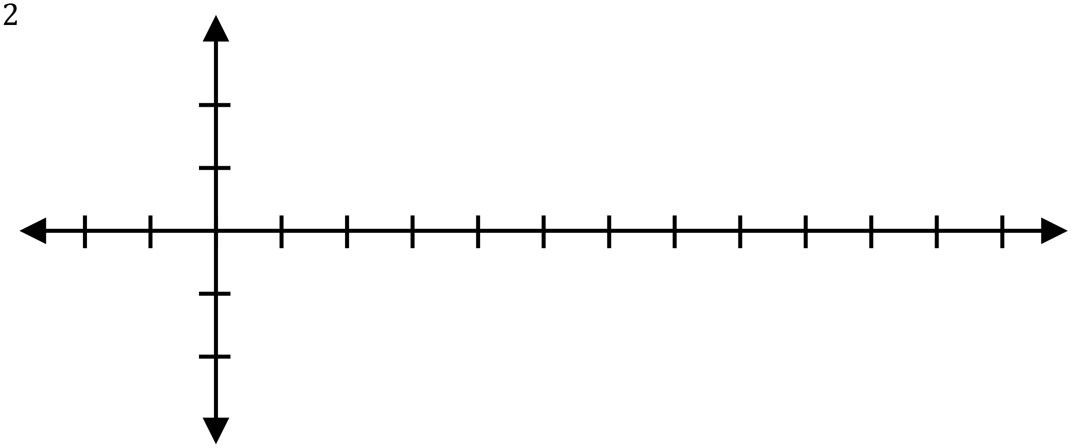


	Period	Domain	Range	Asymptotes	Zeros	Even/ Odd
Tangent						
Cotangent						
Cosecant						
Secant						

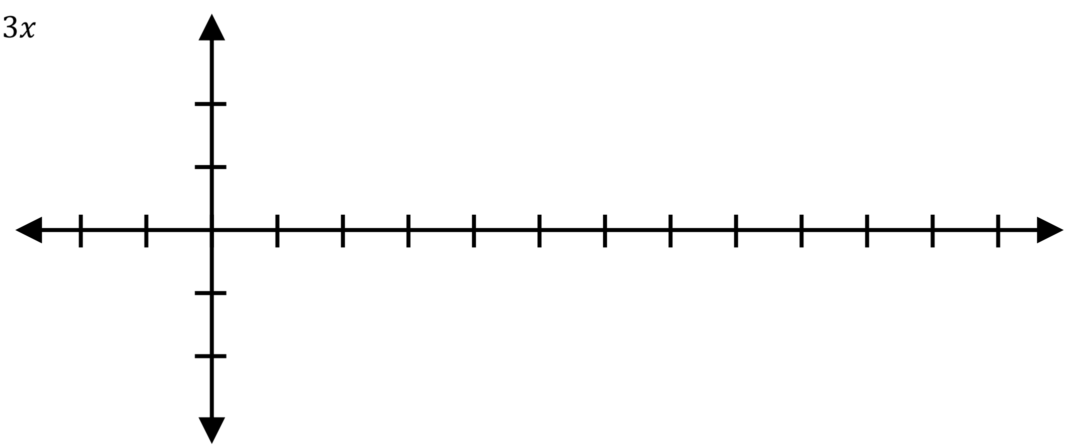
<u>Examples: Graphs of Tangent & Cotangent</u>

Describe the transformations required to obtain the graph of the given function from a basic trigonometric graph. Locate the vertical asymptotes and graph two periods of the function.

1. $y = 3\tan\frac{x}{2}$



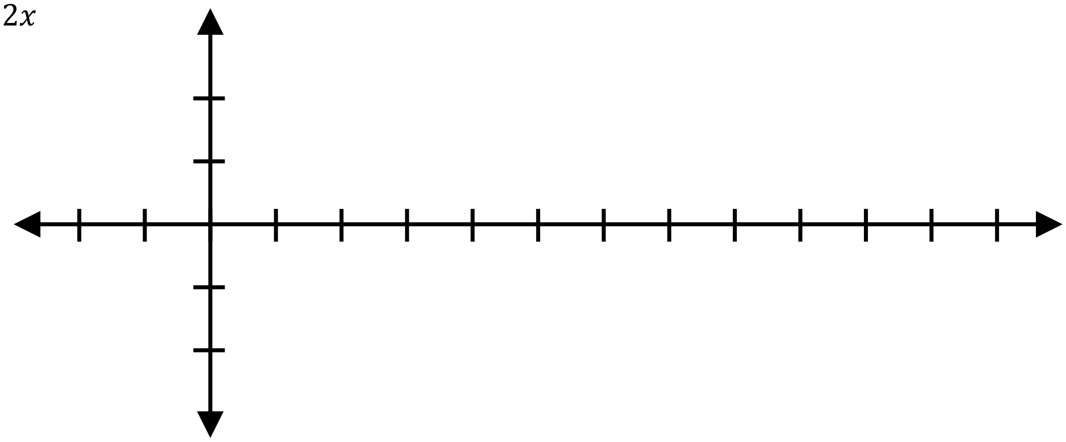
2. $y = -\cot 3x$



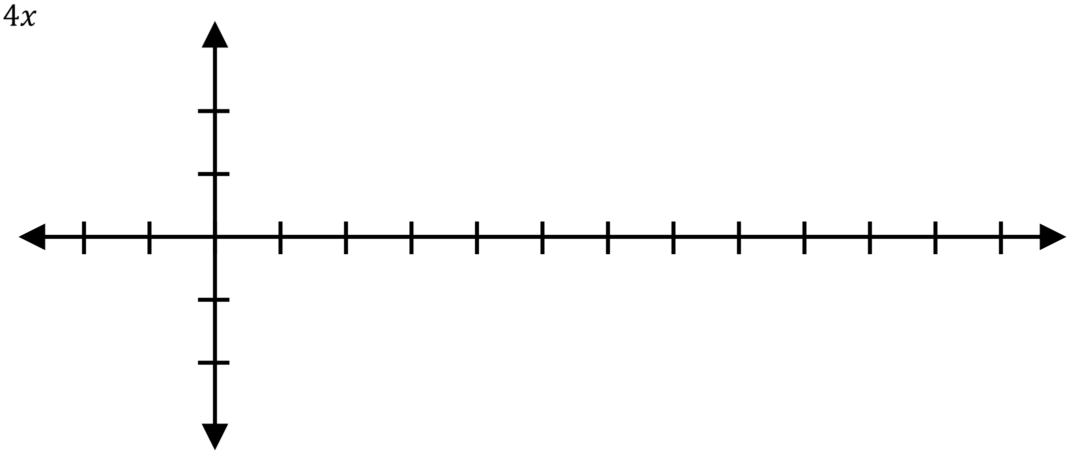
Examples: Graphs of Secant & Cosecant

Describe the transformations required to obtain the graph of the given function from a basic trigonometric graph. Locate the vertical asymptotes and graph two periods of the function.

3. $y = -\csc 2x$



4. $y = 3\sec 4x$



Examples: Transformations of Trig Functions

Describe the transformations required to obtain the graph of the given function from a basic trigonometric graph. (Do not graph.)

5. $y = \sec\left(3x + \frac{\pi}{2}\right)$

6. $y = -\cot\left(\frac{1}{2}x\right) - 1$

Examples: Solving a Trigonometric Equation Algebraically

Solve for x in the given interval. You should be able to find these numbers w/out a calculator, using reference triangles in the proper quadrants.

7. $\sec x = -2;\ \pi \le x \le {}^{3\pi}/_{2}$
8. $\cot x = 1\,;\ -\pi \le x \le -{}^{\pi}/_{2}$

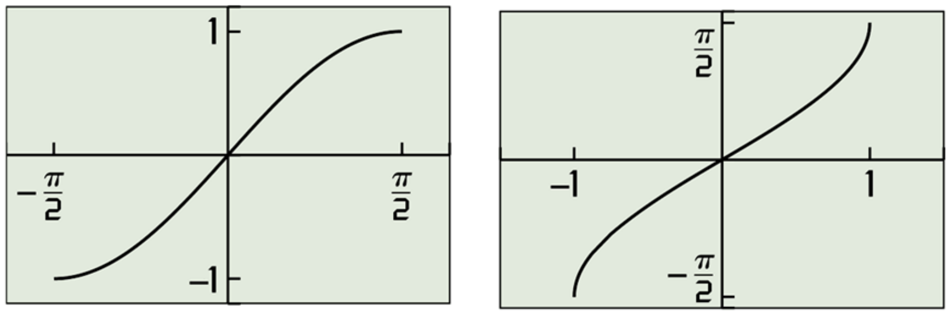
Use a calculator to solve fox x in the given interval.

9. $\csc x = 2; 0 \le x \le \pi$
10. $\tan x = 0.3;\ 0 \le x \le 2\pi$

4.7 INVERSE TRIGONOMETRIC FUNCTIONS

Objectives:

- Arcsine Function: $\mathbf{arcsin}\, \boldsymbol{x} = \mathbf{sin}^{-1} \boldsymbol{x}$



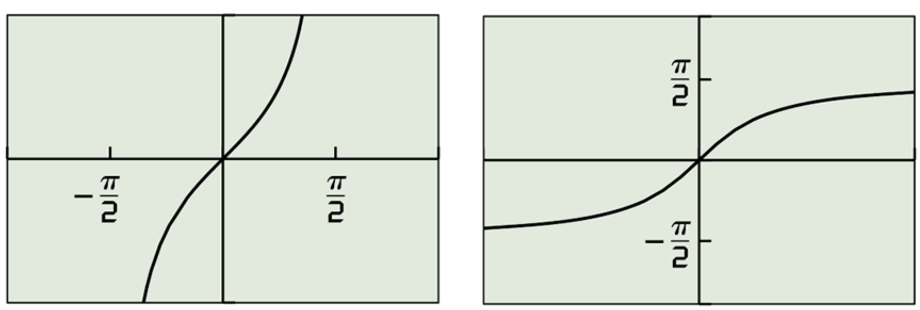
[–2, 2] by [–1.2, 1.2]
(a)

[–1.5, 1.5] by [–1.7, 1.7]
(b)

(a) y = sin x is one-to-one & (b) has an inverse, y = sin^{-1}x

- ➢ The unique angle θ in the interval $\left[-{}^{\pi}/_{2}, {}^{\pi}/_{2}\right]$ such that $\sin y = x$ is the <u>inverse sine</u> (or arcsine) of x.
    - ▪ What's its domain?

- Arctangent Function: $\mathbf{arctan}\, \boldsymbol{x} = \mathbf{tan}^{-1} \boldsymbol{x}$
    - ➢ The unique angle θ in the interval $\left(-{}^{\pi}/_{2}, {}^{\pi}/_{2}\right)$ such that $\tan y = x$ is the <u>inverse tangent</u> (or arctangent) of x.
        - ▪ Domain: Range:

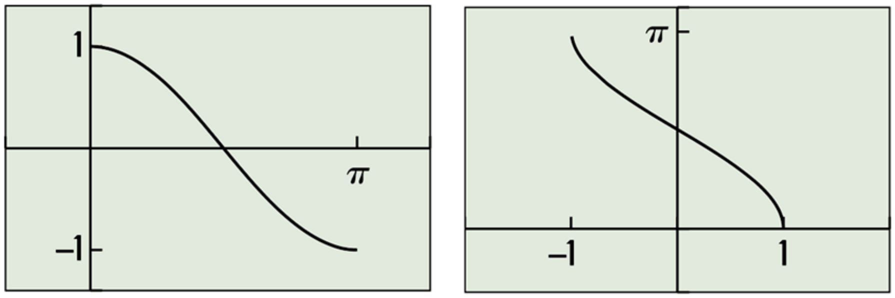


[–3, 3] by [–2, 2]
(a)

[–4, 4] by [–2.8, 2.8]
(b)

(a) y = tan x is one-to-one & (b) has an inverse, y = tan^{-1}x

Arc cosine Function: $\mathbf{arccos}\, x = \mathbf{cos}^{-1} x$



[−1, 4] by [−1.4, 1.4]
(a)

[−2, 2] by [−1, 3.5]
(b)

(a) y = cos x is one-to-one & (b) has an inverse, $y = \cos^{-1} x$

- The unique angle θ in the interval $[0, \pi]$ such that $\cos y = x$ is the inverse cosine (or arccosine) of x.
  - What's its domain?

A little bit more on trig inverse functions:

- Domains of Inverse Trig Functions
  - Sine & tangent – quadrants ____________
  - Cosine – quadrants ____________

- The following equations are always true:
  - $\sin(\sin^{-1} x) = x$
  - $\cos(\cos^{-1} x) = x$
  - $\tan(\tan^{-1} x) = x$

$\text{trig}(\text{trig}^{-1} x) = x$
Work from the inside out.
Solution: an exact value

- The following equations are true for values in the “restricted” domain.
  - $\sin^{-1}(\sin x) = x$
  - $\cos^{-1}(\cos x) = x$
  - $\tan^{-1}(\tan x) = x$

$\text{trig}^{-1}(\text{trig}\, x) = x$
Work from the inside out.
Solution: an angle measure within the respective domain

Examples: Evaluating Inverse Trig Functions

Find the exact value of each expression w/out a calculator. (Make a reference triangle in the proper quadrant.)

1. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
2. $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$
3. $\tan^{-1}\sqrt{3}$
4. $\sin^{-1}\left(\sin\frac{5\pi}{6}\right)$
5. $\tan(\tan^{-1} 1)$
6. $\cos^{-1}\left(\cos\frac{\pi}{3}\right)$
7. $\tan\left(\sin^{-1}\frac{3}{5}\right)$

Use a calculator in to evaluate the expression. Express your answer in both degrees (round to the nearest tenth) and radians (round to the nearest thousandth).

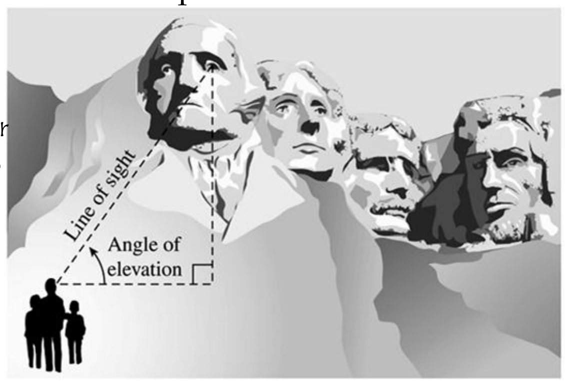
8. $\sin^{-1}(-0.81)$
9. $\cos^{-1} 0.479$
10. $\tan 2.246$

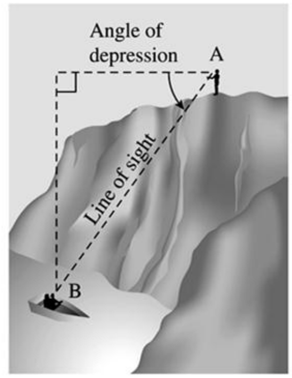
4.8 Solving Problems w/Trigonometry

Objectives:

Angles of Elevation & Depression

Angles of elevation are formed by your line of sight to what you're looking up at.





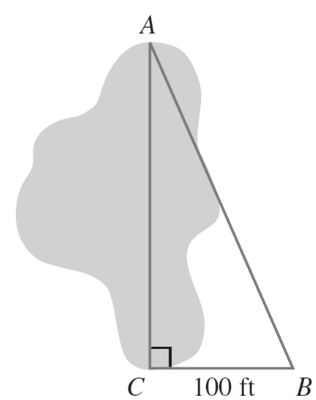
Angles of depression are formed by your line of sight to what you're looking down at.

- The angle of elevation is congruent to the angle of depression.

Examples: Modeling with Sine, Cosine & Tangent

1. The Chrysler Building in New York City was the tallest building in the world at the time it was built. It casts a shadow approximately 130 feet long on the street when the sun's rays form an 82.9° angle with the earth. How tall is the building?

2. Dion's team of surveyors had to find the distance AC across the lake at Montgomery County Park. Field assistants positioned themselves at points A and C while Dion set up an angle-measuring instrument at point B, 100 feet from C in a perpendicular direction. Dion measured $\angle ABC$ as 75°12'42". What is the distance AC?



3. A passenger in an airplane sees two towns directly to the left of the plane.

   a. What is the distance d from the airplane to the first town?
   b. What is the horizontal distance x from the airplane to the first town?
   c. What is the distance between the two towns (y)?