5.1 - Fundamental Identities (Day 1)

<u>Objectives</u>: Use the Fundamental Identities to Simplify Trigonometric Expressions

- Graph the following: $y = \sin^3 x + \cos^2 x \sin x$
  - Make sure you're in <u>radian mode</u> and use "<u>zoom trig</u>"
  - What do you observe?

- What are Trigonometric Identities?
  - A trigonometric equation that is <u>always</u> true throughout its domain

- What are the Basic Trigonometric Identities?
  - Those identities that follow from the definitions of the six basic trig functions

Basic Trigonometric Identities

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Refer to your "Second Semester Summary Sheet" for ALL *Trigonometric Formulas & Identities.*

- Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

  - Use the identity above to derive the remaining Pythagorean Identities:

Divide by $\cos^2\theta$	**Divide by $\sin^2\theta$**

Pythagorean Identities

Example 1: Using the Pythagorean Identities

Evaluate without using a calculator or a reference triangle.

Find $\tan\theta$ & $\csc\theta$ if $\sec\theta = 4$ & $\sin\theta < 0$.

Complementary Angles

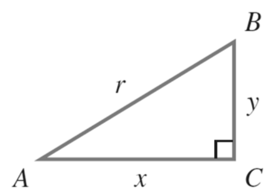


FIGURE 5.2 Angles A and B are complements in right $\triangle ABC$.

- What do you observe when we use the usual triangle ratios to define the six trig functions of angles *A* & *B*?

Angle A:	$\sin A = \dfrac{y}{r}$	$\tan A = \dfrac{y}{x}$	$\sec A = \dfrac{r}{x}$
	$\cos A = \dfrac{x}{r}$	$\cot A = \dfrac{x}{y}$	$\csc A = \dfrac{r}{y}$
Angle B:	$\sin B = \dfrac{x}{r}$	$\tan B = \dfrac{x}{y}$	$\sec B = \dfrac{r}{y}$
	$\cos B = \dfrac{y}{r}$	$\cot B = \dfrac{y}{x}$	$\csc B = \dfrac{r}{x}$

- The value of the function at *A* is the same as the value of its cofunction at *B*.

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta \qquad \cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$$

$$\sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta \qquad \csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$$

Odd-Even Functions

- We know that every basic trigonometric function is either odd or even, ergo:

Odd-Even Identities

$\sin(-x) = -\sin x$	$\cos(-x) = \cos x$	$\tan(-x) = -\tan x$
$\csc(-x) = -\csc x$	$\sec(-x) = \sec x$	$\cot(-x) = -\cot x$

Example 2: Using Cofunction & Odd-Even Identities

Use identities to find the value of the expression:

If $\sin\left(\theta - \frac{\pi}{2}\right) = 0.73$, find $\cos(-\theta)$

Simplifying Trigonometric Expressions

- Use trigonometric identities
- Use Algebra, if necessary:
  - Factor or expand
  - Combine - Rewrite expressions using common denominators

Examples: Simplifying Trigonometric Expressions

Use basic identities to simplify the expression.

3. $y = \sin^3 x + \cos^2 x \sin x$

4. $(\sin x)(\tan x + \cot x)$

5. $\dfrac{1+\tan^2 x}{\csc^2 x}$

Simplify the expression to either a constant or a basic trigonometric function.

6. $\cot(-x)\cot\left(\frac{\pi}{2}-x\right)$

7. $\dfrac{\tan\left(\frac{\pi}{2}-x\right)\csc x}{\csc^2 x}$

Combine the fractions and simplify to a multiple of a power of a basic trigonometric function.

8. $\dfrac{\cos x}{1-\sin x}-\dfrac{\sin x}{\cos x}$

9. $\dfrac{1}{\sin^2 x}+\dfrac{\sec^2 x}{\tan^2 x}$

Write the expression in factored form as an algebraic expression of a single trig function.

10. $\cos x - 2\sin^2 x + 1$

11. $4\tan^2 x - \dfrac{4}{\cot x} + \sin x \csc x$

5.1 - Fundamental Identities (Day 2)

Objectives: Use the Fundamental Identities to Simplify Trigonometric Expressions & Solve Trigonometric Equations

- Solving a Trigonometric Equation
  - Simplify first!
    - Watch for those values that would make both sides of the original equation undefined.

Examples: Solving a Trigonometric Equation

Find all solutions to the equation in the interval $[0, 2\pi)$ without a calculator.

1. $\tan^2 x = 3$
2. $2\cos x \sin x - \cos x = 0$
3. $4\cos^2 x - 4\cos x + 1 = 0$

Examples: Solving a Trigonometric Equation

Find all solutions to the equation using a calculator where needed.

4. $\cos^2 x = 0.4$

5.2 - Proving Trigonometric Identities

Objectives: Determine Whether an Equation is an Identity; Confirm Identities Analytically

When it comes to proving identities, the basic strategies are similar to the rules for forming a "word ladder" AND the path is reversible.

Proving Identities

General Strategies I

1. The proof begins with the expression on one side of the identity.
2. The proof ends with the expression on the other side.
3. The proof in between consists of showing a sequence of expressions, each one easily seen to be equivalent to its preceding expression.

Example 1: Prove an Identity

$$(1 - \tan x)^2 = \sec^2 x - 2\tan x$$

General Strategies II

1. Begin with the more complicated expression and work toward the less complicated expression.
2. If no other move suggests itself, convert the entire expression to one involving sines and cosines.
3. Combine fractions by combining them over a common denominator.

More Examples: Prove the Identity

2. $$\frac{(1-\cos u)(1+\cos u)}{\cos^2 u} = \tan^2 u$$

3. $$\frac{\sin\theta}{1-\cos\theta} + \frac{1+\cos\theta}{\sin\theta} = \frac{2(1+\cos\theta)}{\sin\theta}$$

General Strategies III

1. Use the algebraic identity $(a+b)(a-b) = a^2 - b^2$ to set up applications of the Pythagorean identities.
2. Always be mindful of the "target" expression, and favor manipulations that bring you closer to your goal.

More Examples: Prove the Identity

4. $\sin^2 x \cos^3 x = (\sin^2 x - \sin^4 x)(\cos x)$

5. $\sin^5 x = (1 - 2\cos^2 x + \cos^4 x)(\sin x)$

5.3 - Sum & Difference Identities

Objective: Apply the Identities for the Sine, Cosine or Tangent of a Sum or Difference

Cosine: Sum & Difference Identities

Cosine of a Sum or Difference

$$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$$

(Note the sign switch in either case.)

Example 1: Using Cosine of a Sum or Difference

Find the exact value: $\cos\frac{\pi}{12}$

Sine: Sum & Difference Identities

Sine of a Sum or Difference

$$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$$

(Note that the sign does *not* switch in either case.)

Example 2: Using Sine of a Sum or Difference

Find the exact value: $\sin 15°$

Tangent: Sum & Difference Identities

$$\boxed{\tan(u \pm v) = \frac{\tan u \pm \tan v}{1 \mp \tan u \tan v}}$$

Example 3: Using Tangent of a Sum or Difference

Find the exact value: $\tan\frac{5\pi}{12}$

Examples: Proving Identities

4. $\tan\left(\theta + \frac{\pi}{4}\right) = \frac{1 + \tan\theta}{1 - \tan\theta}$

5. $\sin\left(x+\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}\sin x+\frac{1}{2}\cos x$

6. $\cos 3x=\cos^3 x-3\sin^2 x\cos x$

5.4 - Multiple-Angle Identities

Objective: Apply the Double-Angle, Power-Reducing and Half-Angle Identities

- Double-Angle Identities
  - The result of letting u = v in any of the angle sum identities.

$$\sin 2u = 2\sin u\cos u$$

$$\cos 2u = \begin{cases}\cos^2 u-\sin^2 u\\ 2\cos^2 u-1\\ 1-2\sin^2 u\end{cases}$$

$$\tan 2u=\frac{2\tan u}{1-\tan^2 u}$$

Example 1: Proving a Double-Angle Identity

$\cos 2u = \cos^2 u - \sin^2 u$

Examples: Using Double-Angle Identities

Use double-angle identities to simplify the expression and then solve it algebraically for exact solutions in the interval $[0, 2\pi)$.

2. $\sin 2x - 2\sin x = 0$
3. $\cos 2x - \cos x = 0$
4. $\cos 2x = \sin x$

Example 5: Using Double-Angle Identities

Write the expression as one involving sine and/or cosine of one angle:

$\sin 2\theta + \cos 3\theta$

Half-Angle Identities

- Caution!
  - There is an unavoidable ambiguity of sign involved w/the square root that must be resolved in cases of sine and cosine by checking the quadrant in which the half-angle lies.

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}}$$

$$\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$$

$$\tan\frac{u}{2} = \begin{cases} \pm\sqrt{\dfrac{1-\cos u}{1+\cos u}} \\ \dfrac{1-\cos u}{\sin u} \\ \dfrac{\sin u}{1+\cos u} \end{cases}$$

<u>Examples: Using Half-Angle Identities</u>

Use half-angle identities to find an exact value without a calculator.

6. $\cos 75°$

7. $\tan\dfrac{7\pi}{12}$

∿∿

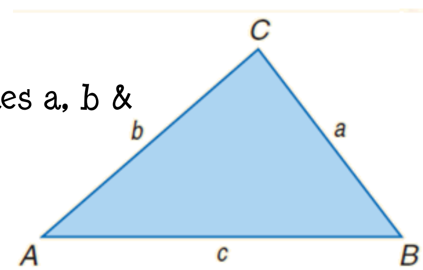
5.5 - The Law of Sines

<u>Objective:</u> Use the Computational Applications of the Law of Sines to Solve a Variety of Problems

The Law of Sines

- The ratio of the sine of an angle to the length of its opposite side is the same for all three angles of any triangle.

- In any △ABC, with angles A, B & C opposite sides a, b & c the following equation is true:



Solving Triangles (AAS, ASA)

- Two angles & a side of a triangle, in any order, determine the size & shape of a triangle completely.

Example: Solving a Triangle Given Two Angles & a Side

Solve $\triangle ABC$. (Round angles to the nearest tenth & sides to the nearest hundredth.)

1. $\angle A = 68°$; $\angle B = 15°$; & $a = 34$

$\angle C$ ____________

b ____________

c ____________

2. $\angle B = 100°$; $\angle C = 20°$; & $b = 18$

$\angle A$ ____________

a ____________

c ____________

The Ambiguous Case (SSA)

- Two sides & an angle are sometimes or never sufficient to determine size & shape of a triangle.
  - If the angle is opposite one of the sides—the SSA case—then there **might** be one, two or zero triangles determined.

Schultz says:
When you find yourself face-to-face with the "ambiguous case" solve as if there ARE two triangles. You'll know soon enough, using subtraction, just how many triangles exist with such measures.

Examples: Solving a Triangle Given Two Sides & an Angle

Solve $\triangle ABC$, if possible. (Round angles to the nearest 10th & sides to the nearest 100th)

3. $\angle A = 61^\circ$; $a = 8$ & $b = 21$

$\angle B$ ____________

$\angle C$ ____________

c ____________

4. $\angle B = 75^\circ$; $b = 48$ & $c = 49$

$\angle C$ ____________

$\angle A$ ____________

a ____________

5. $\angle C = 70^\circ$; $b = 9$ & $c = 14$

$\angle B$ ____________

$\angle A$ ____________

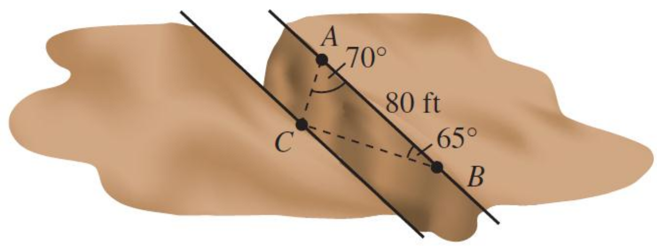
A ____________

Applications Involving the Law of Sines

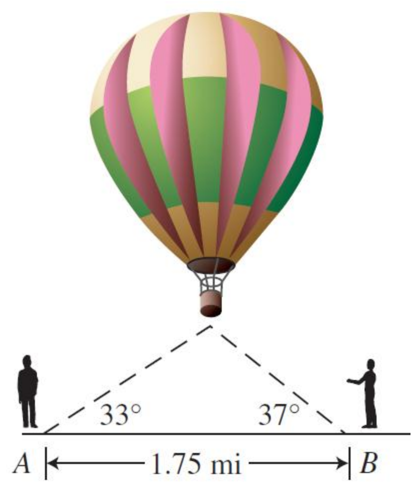
- Many problems involving angles and distances can be solved by superimposing a triangle onto the situation and solving the triangle.

Examples: Applications w/Oblique Triangles

6. Two markers A & B on the same side of a canyon rim are 80 feet apart, as shown in the figure. A hiker is located across the rim at point C. A surveyor determines that $\angle BAC = 70°$ and $\angle ABC = 65°$.
   a) What is the distance between the hiker and point A?
   b) What is the distance between the two canyon rims? (Assume they are parallel.)



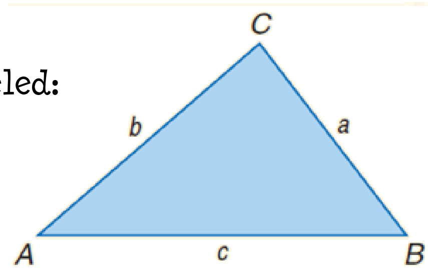
7. A hot-air balloon is seen over Tucson, Arizona, simultaneously by two observers at points A and B that are 1.75 miles apart on level ground and in line with the balloon. The angles of elevation are as shown here. How high above the ground is the balloon?



5.6 - The Law of Cosines

Objectives: Apply the Law of Cosines to Solve Acute & Obtuse Triangles and Determine the Area of a Triangle in Terms of the Measures of its Sides & Angles

- The Law of Cosines
  - AKA the "generalized Pythagorean Theorem"
  - Let △ABC be any triangle w/sides & angles labeled:
    - Then…



- Solving Triangles (SAS & SSS)
  - Side-Angle-Side Triangles
    - Start by finding the third side using the L.o.C.
    - Either law can be used to find one of the two remaining angles
      - It is better to use the L.o.C. to find angles since the arccosine function will distinguish obtuse angles from acute angles
  - Side-Side-Side Triangles
    - A triangle with side lengths a, b and c, exists if…

$$a + b > c \text{ and } b + c > a \text{ and } a + c > b$$

    - Use the L.o.C. to find two of the three angles
      - Do not use the L.o.S.

Examples: Solving Triangles (SAS)

Round sides to the nearest 100^{th} and angles to the nearest 10^{th}.

1. a = 15, b = 7 & ∠C = 40°

c ____________

∠A ____________

∠B ____________

2. $a = 5$, $c = 6$ & $\angle B = 112°$

b ____________

$\angle A$ ____________

$\angle C$ ____________

Examples: Solving Triangles (SSS)

Round sides to the nearest 100^{th} and angles to the nearest 10^{th}.

3. $a = 7$, $b = 14$ & $c = 15$

$\angle A$ ____________

$\angle B$ ____________

$\angle C$ ____________

4. $a = 24$, $b = 20$ & $c = 16$

$\angle A$ ____________

$\angle B$ ____________

$\angle C$ ____________

Schultz says:

Since we primarily rely on our graphers to come up with the accurate calculations, be sure to show in your work what a (or ∠A) is exactly.

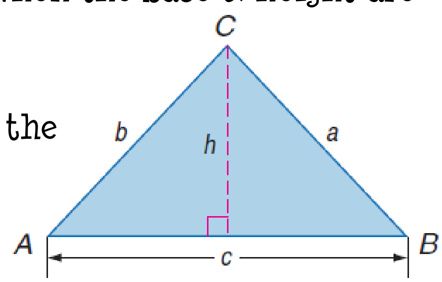
$$a = \frac{14.5 \sin 52^\circ}{\sin 28^\circ}$$
$$a \approx 24.3$$

$$\cos A = \frac{5^2 + (6.529\ldots)^2 - 11^2}{2(5)(6.529\ldots)}$$
$$A = \cos^{-1}\left(\frac{5^2 + (6.529\ldots)^2 - 11^2}{2(5)(6.529\ldots)}\right)$$
$$\approx 144.8^\circ$$

Teachers like this kind of stuff.

Finding the Area of a Triangle

- You know how to find the area of a triangle when the base & height are known.
  - $A_\triangle = \frac{1}{2}bh$
    - What would the area of the triangle (to the right) be via the formula?



- What if the height of the triangle was not known?
  - How can we use ∠A to find the height of △ABC?

- Going back to the basic formula above, if we make the substitution (we just found) for *h*, then we have an alternate formula for finding the area of a triangle:
  - $A_\triangle$ = ________________________

<u>Example: Finding the Area of a Triangle</u>

5. ∠B = 101°; a = 10 cm & c = 22 cm

Finding the Area of a Triangle (SSS)

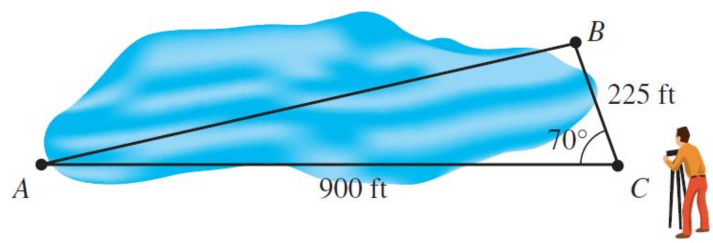
- Heron's Formula
  - Determine whether a triangle with side lengths a, b and c, exists:
    $$a + b > c \text{ and } b + c > a \text{ and } a + c > b$$
  - Let a, b & c be the sides of $\triangle ABC$, and let s denote the <u>semi-perimeter</u>, then the area of $\triangle ABC$ is given by:
    $$s = \frac{a + b + c}{2} \; \& \; A = \sqrt{s(s-a)(s-b)(s-c)}$$

<u>Example: Finding Area w/Heron's Formula</u>

6. Find the area of a triangle with the sides: 13, 17, 20

<u>Examples: Applications Involving the Law of Cosines</u>

7. In order to determine the distance between two points A and B on opposite sides of a lake, a surveyor chooses a point C that is 900 feet from A and 225 feet from B, as shown. If the measure of the angle at C is 70°, find the distance between A and B.



8. Two boats start at the same point and speed away along courses that form a 110° angle. If one boat travels at 24 miles per hour and the other boat travels at 32 miles per hour, how far apart are the boats after 30 minutes? (Draw a diagram.)