Name: ______________________________

Chapter 8: 3D Figures

8.1 THREE-DIMENSIONAL FIGURES

Objectives:

- Classify solids
- Determine the shapes of cross sections and intersections of solids & planes.

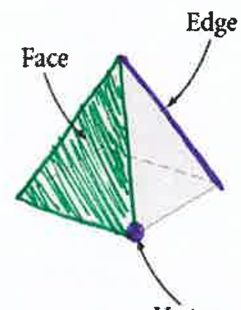
❖ Types of Solids

Polyhedra | **Not Polyhedra**

❖ **Polyhedron**

➢ A solid formed by polygons that enclose a single region of space.

- The flat polygonal surfaces of a polyhedron are called its faces.
- A segment where two faces intersect is called an edge.
- The point of intersection of three or more edges is called a vertex of the polyhedron.

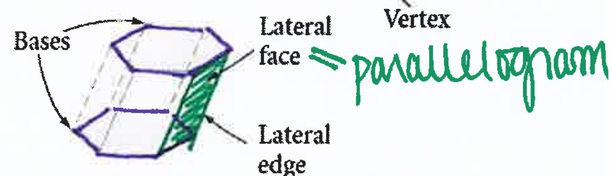


❖ **Prisms**

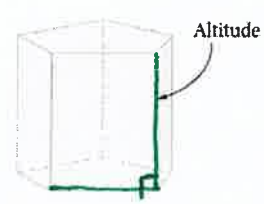
➢ A prism is a special type of polyhedron, with two faces called bases, which are congruent, parallel polygons.

- The other faces of the polyhedron, called lateral faces, are parallelograms that connect the corresponding sides of the bases.
- The lateral faces meet to form lateral edges.

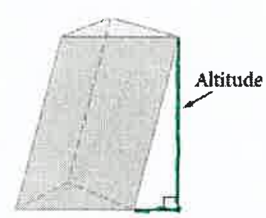
  Prisms are classified by their bases.
- A prism whose lateral faces are rectangles is called a right prism. Its lateral edges are perpendicular to its bases.
- A prism that is not a right prism is called an oblique prism.
- The altitude (aka height) of a prism is any perpendicular segment from one base to the place of the other base.



Hexagonal prism



Right pentagonal prism



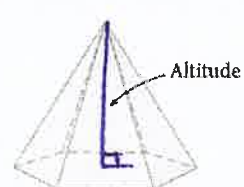
Oblique triangular prism

Base

❖ **Pyramids**

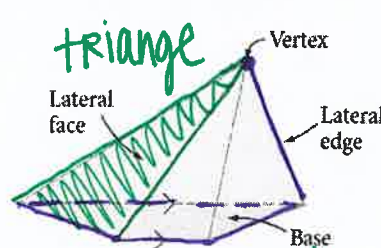
➢ A pyramid is a polyhedron with one polygonal face (called the base) and whose lateral faces are triangles formed by segments connecting the vertices of the base to a common point, the vertex, not on the base.

- Pyramids are classified by their bases.
- The altitude (aka height) of a pyramid is the perpendicular segment from its vertex to the place of its base.



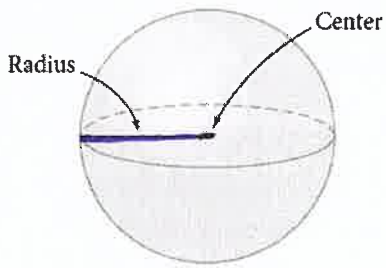
Hexagonal pyramid

Base

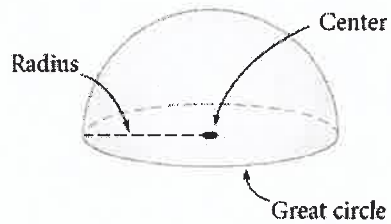


Trapezoidal pyramid

- **Circular Solids**
- **Spheres**
  - A sphere is the set of all points in space at a given distance from a given point.
    - The given distance is called the radius of the sphere, and the given point is the center of the sphere.
  - A hemisphere is half a sphere and its circular base.

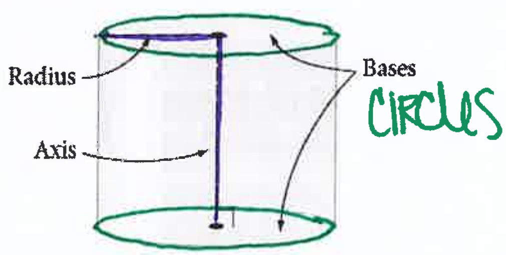


Sphere

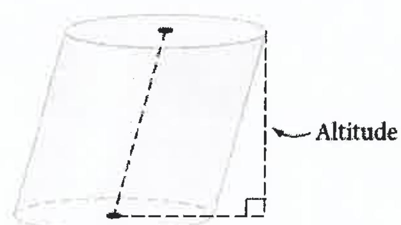


Hemisphere

- **Cylinders**
  - A cylinder is a solid with two bases which are congruent, parallel circles.
    - The segment connecting the centers of the bases is called the axis of the cylinder.
    - The radius of the cylinder is the radius of a base.
    - If the axis of a cylinder is perpendicular to the bases, then the cylinder is a right cylinder.
    - A cylinder that is not a right cylinder is an oblique cylinder.
    - The altitude (aka height) of a cylinder is any perpendicular segment from the plane of one base to the plane of the other.

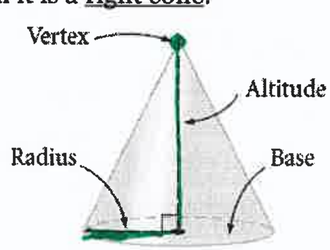


Right cylinder

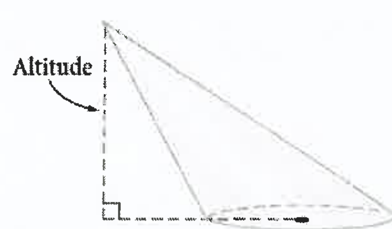


Oblique cylinder

- **Cones**
  - A cone has a circular base and a vertex.
    - The radius of a cone is the radius of its base.
    - The vertex of a cone is the point that is the greatest perpendicular distance from the base.
    - The altitude (aka height) is the perpendicular segment from the vertex to the place of the base.
    - If the line segment connecting the vertex of a cone with the center of its base is perpendicular to the base, then it is a right cone.



Right cone



Oblique cone

- Investigating Cross Sections
  - A cross section of a solid is the two-dimensional figure formed by the intersection of a plane and a solid when a plane passes through the solid.
  - Go to www.geogebra.org
    - Search for: **3D Geometry & Cross Sections**
      Author: Duke
    - Click on *Cross Sections & Solids for FSA(7)*
    - Explore the sections (listed below) and record your finds in the table (below).

Solid	*Describe the cross section(s) formed when a plane intersects the solid.*
Sections of Rectangular Pyramids	*3 total* Rectangle (horizontal) general quadrilateral ← slant trapezoid triangle (vertical)
Sections of Cylinders	*3 total* (horizontal / parallel to base) circle ellipse – slant rectangle (vertical / $\perp$ to base)
Sections of Cones	*2 total* circle // to base ellipse slant triangle $\perp$ to base
Sections of Spheres	*1 total* circle
Sections of Cubes	*5 total* square rectangle triangle pentagon hexagon

8.2 TRANSFORMING TWO-DIMENSIONAL FIGURES

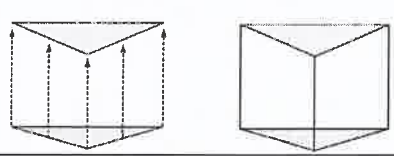
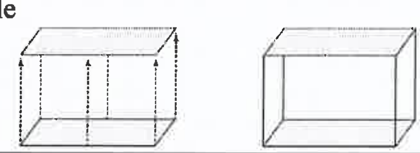
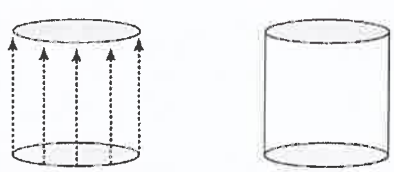
Objectives:

- Apply rotations & translations to two-dimensional plane figures to create three-dimensional solids.
- Describe three-dimensional solids formed by rotations & translations of plane figures through space.
- Build three-dimensional solids by stacking congruent or similar two-dimensional plane figures.

❖ Investigating Rotations in Three-Dimensional Space
- ➢ Go to www.geogebra.org
  - Search for: **Rotating 2D Shapes to Make 3D Shapes**
    Author: Sobarrera
  - Explore rotating triangles, rectangles and circles.
  - Record your findings below.

2D Shape	3D Shape	*Relate the dimensions of the 2D shape to the 3D shape*
Triangle	cone	height, base; height (same), radius = base
Rectangle	cylinder	h, b; height (same), radius = base
Circle	sphere	r; same radius

❖ Translations in Three-Dimensional Space
- ➢ A 2D figure is translated through space in a direction that is perpendicular to the plane containing the 2D figure to form the solid shown:

2D Shape	3D Shape
Triangle 	triangular prism
Rectangle 	rectangular prism
Circle 	cylinder

- Stacking Two-Dimensional Congruent or Similar Figures
  - Stacking Circles

    - What is the name of the solid formed by a stack of congruent circles?

      cylinder

    - What is the name of the solid formed by a stack of similar circles?

      cone

  - Stacking Squares
    - What is the name of the solid formed by a stack of congruent squares? rectangular prism

    - What is the name of the solid formed by a stack of similar squares?

      square pyramid

  - Stacking Triangles
    - What is the name of the solid formed by a stack of congruent triangles?

      triangular prism

    - What is the name of the solid formed by a stack of similar triangles?

      triangular pyramid

8.3 VOLUME OF PRISMS & CYLINDERS

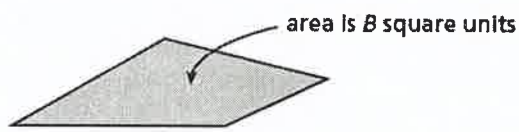
Objectives:

- Use Cavalieri's principles to estimate the approximate volume of irregular or oblique figures.
- Calculate the volume of prisms and cylinders

Recall that the *volume* of a three-dimensional figure is the number of non-overlapping cubic units contained in the interior of the figure.

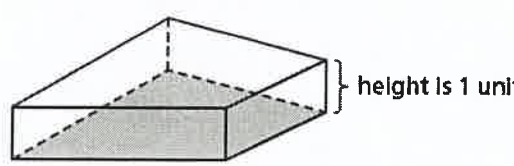
PROBLEM 1 ~ Developing a Basic Volume Formula

A Consider a figure that is the base of a prism or cylinder. Assume the figure has an area of B square units.



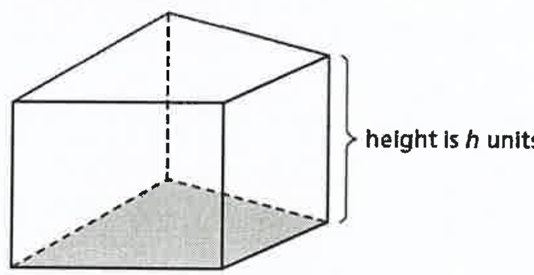
B Use the base to build a prism or cylinder with height 1 unit.

This means the prism or cylinder contains $1 \cdot B$ cubic units.



C Now use the base to build a prism or cylinder with a height of h units.

So, the volume of the prism or cylinder is $B \cdot h$ cubic units.

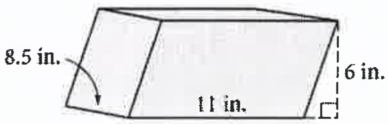


Complete the Conjecture:

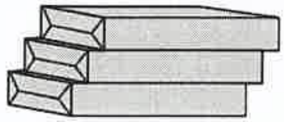
If B is the area of the base of a right prism and h is the height, then the formula for volume is $\underline{Bh}$.

Volume of Oblique Prisms

What about the volume of an oblique prism or cylinder? You can approximate the shape of this oblique rectangular prism with a staggered stack of three reams of 8.5-by-11-inch paper. If you nudge the individual pieces of paper into a slanted stack, then your approximation can be even better.



Oblique rectangular prism


Stacked reams of 8.5-by-11-inch paper

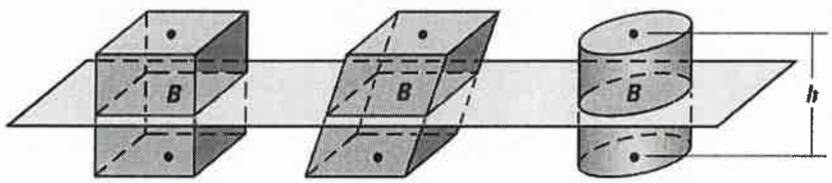

Stacked sheets of paper


Sheets of paper stacked straight

Rearranging the paper into a right rectangular prism changes the shape, but certainly the volume of paper hasn't changed. The area of the base, 8.5 by 11 inches, didn't change and the height, 6 inches, didn't change, either.

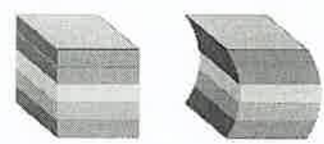
❖ Cavalieri's Principle for Volume

- Consider the solids below. All three have cross sections with equal areas, B, and all three have equal heights, h. By Cavalieri's Principle, it follows that each solid has the same volume.



EXAMPLE 1 ~ Cavalieri's Principle

Each stack of memo papers shown contains 500 sheets of paper. Explain why the stacks have the same volume. Then calculate the volume, given that each sheet of paper is 3 inches by 3 inches by 0.01 inches.



Cross sects. have equal areas

Each sheet of paper has the same area $= 3in \cdot 3in = 9in^2 = B$

Each stack has the same height $= 500 \cdot 0.01in = 5in$ h

therefore the volume is the same $= 9in^2 \cdot 5in = 45in^3$

$B \cdot h = V$

EXAMPLE 2 ~ Volume Applications

A cord of firewood is 128 cubic feet. Macy has three storage boxes for firewood that each measure 2 feet by 3 feet by 4 feet. Does she have enough space to order a full cord of firewood? A half cord? A quarter cord? Explain your reasoning.

$V_{BOX} = 2 \cdot 3 \cdot 4 = 24ft^3 \times 3\text{ boxes} = 72ft^3$

not enough for a full cord

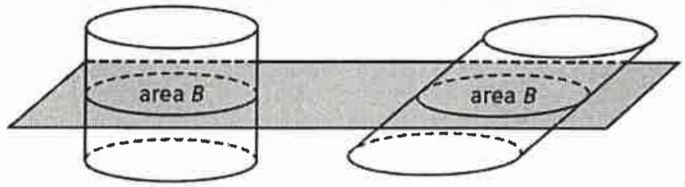
* $\frac{1}{2}$ cord $= 64ft^3$ *

there's enough for a half cord

Volume of Cylinders

Cavalieri's Principle

If two solids have the same height and the same cross-sectional area at every level, then the two solids have the same volume.

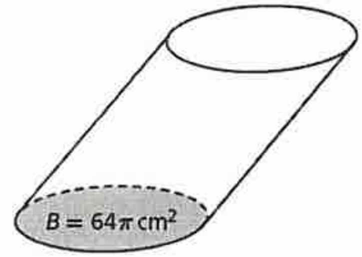


You can think of any oblique cylinder as a right cylinder that has been "pushed over" so that the cross sections at every level have equal areas. By Cavalieri's principle, the volume of an oblique cylinder is equal to the volume of the associated right cylinder. This means the formula $V = Bh = \pi r^2 h$ works for any cylinder.

EXAMPLE 3 ~ Finding the Volume of an Oblique Cylinder

a. Find the height of the cylinder shown if the height is twice the radius.

b. Find the volume of the cylinder.



$B = \pi R^2$
$64\cancel{\pi} = \cancel{\pi} R^2$
$\sqrt{64} = \sqrt{R^2}$
$8 = R$

$h = 2R$
$h = 2(8)$
$h = 16$

$V = Bh$
$V = 64\pi \cdot 16$
$V = 1024\pi \text{ cm}^3$
$V \approx 3217 \text{ cm}^3$

EXAMPLE 4 ~ Comparing Densities

You gather data about two wood logs that are approximately cylindrical. Based on the data in the table, which wood is denser, Douglas fir or American redwood?

$$\text{Density} = \frac{\text{weight}}{\text{volume}}$$

Type of Wood	Diameter (ft)	Height (ft)	Weight (lb)
Douglas fir	1 R=0.5	6	155.5
American redwood	3 R=1.5	4	791.7

DF
$V = \pi(0.5)^2 \cdot 6$
$V = 1.5\pi \text{ ft}^3$
$D = \frac{155.5}{1.5\pi} \approx 33 \text{ lb/ft}^3$

AR
$V = \pi(1.5)^2 \cdot 4$
$V = 9\pi \text{ ft}^3$
$D = \frac{791.7}{9\pi} \approx 28 \text{ lb/ft}^3$

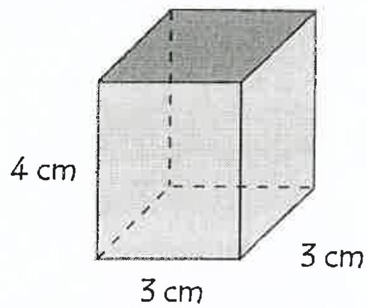
Douglas Fir is more dense

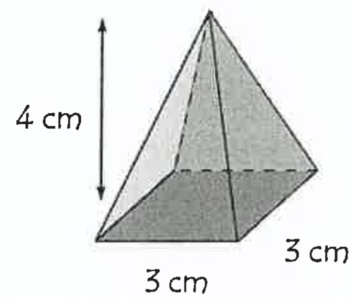
8.4 VOLUME OF PYRAMIDS & CONES

Objective:

- Calculate the volume of pyramids and cones

PROBLEM 1 ~ Connecting the Volumes of Prisms and Pyramids





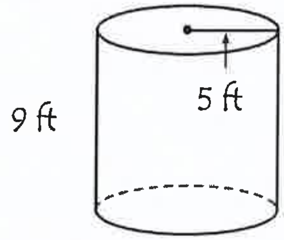
a. How are the solids alike? Square base w/ area = 9 cm^3
same height = 4 cm

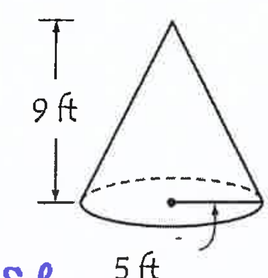
b. How are they different?
prism vs. pyramid

c. Find the volume of the rectangular prism. $V = 9 \cdot 4 = 36$ cm^3

d. How does the volume of the prism compare to that of the rectangular pyramid, if the pyramid's volume is 12 cubic centimeters?
$36 \div 3 = 12$
the pyramid is 1/3 the volume of the prism

PROBLEM 2 ~ Connecting the Volumes of Cylinders and Cones



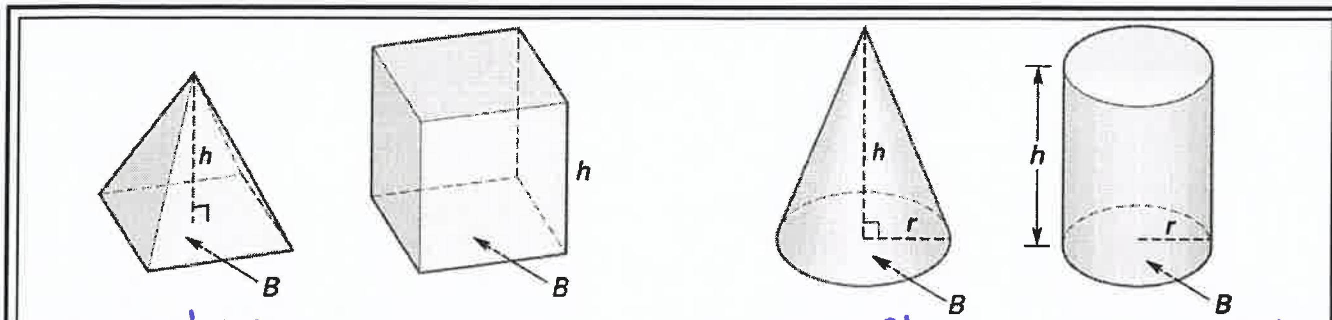


a. How are the solids alike? Circular base
base area = 25π ft^2 same height = 9 ft

b. How are they different?
cylinder vs. cone

c. Find the volume of the cylinder. *Leave your answer in terms of π.*
$V = 25\pi \cdot 9 = 225\pi$ ft^3

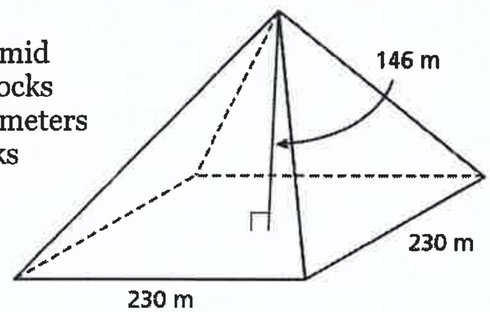
d. How does the volume of the cylinder compare to that of the cone, if the cone's volume is 75π cubic feet?
$225\pi \div 75\pi = 3$
the cone is 1/3 the volume of the cylinder



$V_{\text{pyramid}} = \frac{1}{3}Bh$ $V_{\text{prism}} = Bh$ $V_{\text{cone}} = \frac{1}{3}\pi R^2 h$ $V_{\text{cylinder}} = \pi R^2 h$

EXAMPLE 1 ~ Volume Application with Pyramids

The Great Pyramid in Giza, Egypt, is approximately a square pyramid with the dimensions shown. The pyramid is composed of stone blocks that are rectangular prisms. An average block has dimensions 1.3 meters by 1.3 meters by 0.7 meters. Approximately how many stone blocks were used to build the pyramid?



$V_{pyr} = \frac{1}{3}(230)^2 \cdot 146 = \frac{7723400}{3}$

$\approx 2{,}574{,}467 \text{ m}^3$

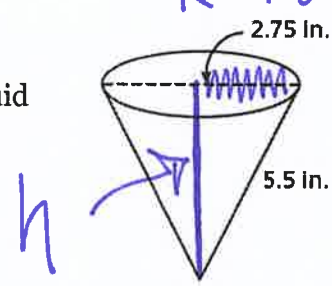
$V_{Block} = (1.3)^2 \cdot 0.7 = 1.183 \text{ m}^3$

$\#\text{ of blocks} = \frac{V_{pyr}}{V_{Block}} = 2176219 \text{ blocks}$

EXAMPLE 2 ~ Volume Application with Cones

A conical paper cup has the dimensions shown. How many fluid ounces of liquid does the cup hold? Round to the nearest tenth. (*Hint:* 1 in.^{3} ≈ 0.554 fl oz.)

$R = 1.375$



$V_{cup} = \frac{1}{3}\pi(1.375)^2(5.33)$

$\approx 10.55 \text{ in}^3$

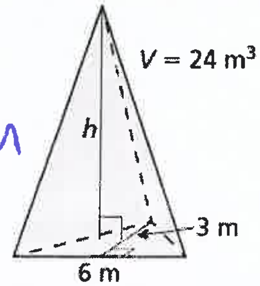
$h^2 + (1.375)^2 = 5.5^2$

$h \approx 5.33 \text{ in}$

$10.55 \text{ in}^3 \cdot \frac{0.554 \text{ fl. oz}}{1 \text{ in}^3} = 5.8$ fluid ounces

EXAMPLES ~ Volume of Pyramids and Cones

3. Find the height of the triangular pyramid.



triangular base

$V = \frac{1}{3}Bh$

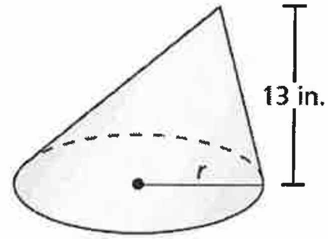
$24 = \frac{1}{3}(9)h$

$24 = 3h$

$8 = h$ meters

$B = \frac{1}{2}(6)(3) = 9$

4. Find the radius of the cone.



$V = 351\pi$ in.^{3}

$V = \frac{1}{3}\pi R^2 h$

$\frac{351\cancel{\pi}}{13} = \frac{\frac{1}{3}\cancel{\pi}R^2 \cdot 13}{13}$

$3 \cdot 27 = \frac{1}{3}R^2 \cdot 3$

$\sqrt{81} = \sqrt{R^2}$

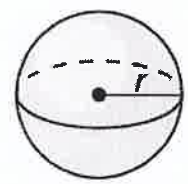
9 in. $= R$

8.5 VOLUME OF A SPHERE

Objective:

- Calculate the volume of spheres

Recall that a *sphere* is the set of points in space that are a fixed distance from a point called the *center* of the sphere. The intersection of a sphere and a plane that contains the center of the sphere is a *great circle*. A great circle divides a sphere into to congruent halves that are called *hemispheres*.



$V = \frac{4}{3}\pi r^3$

EXAMPLES ~ Volume of a Sphere

Find the volume of the ball shown. If necessary, round to the nearest hundredth.

1. Bowling ball

$d = 8.5$ in.

$R = 4.25$ in.

$V = \frac{4}{3}\pi(4.25)^3$

$V \approx 321.56$ in^3

2. Basketball

$C = \underline{29.5}$ in. $= 2\pi R$

$\frac{29.5}{2\pi} = \frac{2\pi R}{2\pi}$

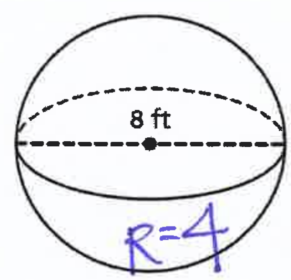
$4.70 = R$

$V = \frac{4}{3}\pi(4.70)^3$

$V \approx 434.89$ in^3

EXAMPLE 3 ~ Volume Application with Spheres

A British thermal unit (BTU) is a unit of energy. It is approximately the amount of energy needed to increase the temperature of one pound of water by one degree Fahrenheit. As you will see in the following example, the energy content of a fuel may be measured in BTUs per unit of volume.



A spherical gas tank has the dimensions shown. When filled with natural gas, it provides 275,321 BTU. How many BTUs does one cubic foot of natural gas yield? Round to the nearest BTU.

$V = \frac{4}{3}\pi(4)^3 = \frac{256}{3}\pi \approx 268.08$ ft^3

$\frac{268.08 \text{ ft}^3}{1 \text{ ft}^3} = \frac{275321 \text{ BTU}}{?}$

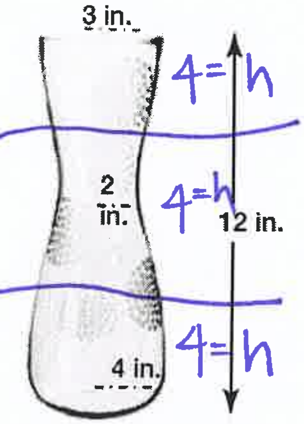
1027.01 BTUs

8.6 APPLICATIONS WITH VOLUME

Objectives:

- Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.
- Use geometric shapes, their measures, and their properties to describe objects
- Apply concepts of density based on area and volume in modeling situations
- Apply geometric methods to solve design problems

1. Describe a strategy for approximating the volume of the vase shown. Then determine the approximate volume of the vase to the nearest hundredth.



3 cylinders, each w/ a height = 4in

$V_{cyl_1} = \pi \cdot 3^2 \cdot 4 = 36\pi$

$V_{cyl_2} = \pi \cdot 2^2 \cdot 4 = 8\pi$

$V_{cyl_3} = \pi \cdot 4^2 \cdot 4 = 64\pi$

$+$

$108\pi \approx 339.29 \text{ in}^3$

2. A typical hot-air balloon is about 75 feet tall and about 55 feet in diameter at its widest point. About how many cubic feet of hot air does a typical hot-air balloon hold? Explain how you determined your answer.

$V_{hemi} = \frac{1}{2} \cdot \frac{4}{3} \cdot \pi (27.5)^3$

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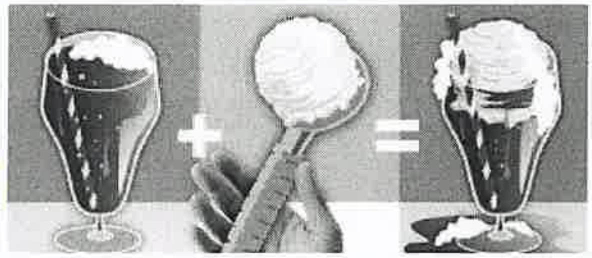
$V_{cone} = \pi (27.5)^2 (47.5)$

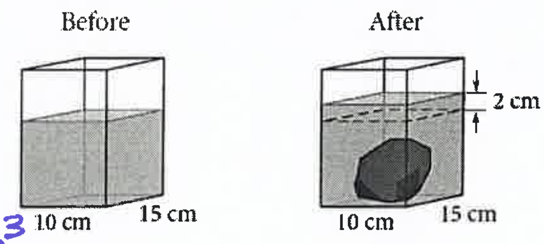
156408.77 ft^3

Displacement



What happens if you add a scoop of ice cream to a glass filled with root beer?

The volume of the liquid that overflows equals the volume of the solid below the liquid level. This volume is called the object's displacement.



3. Marlie wants to find the volume of an irregularly shaped rock. She puts some water into a rectangular prism with a base that measures 10 cm by 15 cm. When the rock is put into the contain, Marlie notices that the water level rises 2 cm because the rock displaces its volume of water. What is the volume of the rock?

$V_{ROCK} = 10 \cdot 15 \cdot 2 = 300 \text{ cm}^3$

Density

An important property of a material is density. Density is the mass of matter in a given volume. You can find the mass of an object by weighing it. You can calcuate the density by dividing the mass by the volume:

$$\text{density} = \frac{\text{mass}}{\text{volume}}$$

4. A clump of metal weighing 351.4 grams is dropped into a cylindrical container, causing the water level to rise 1.1 cm. The radius of the base of the container is 3 cm. What is the density of the metal? Given the table, and assuming the metal is pure, what is the metal?

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Metal	Density	Metal	Density
Aluminum	2.81 g/cm³	Nickel	8.89 g/cm³
Copper	8.97 g/cm³	Platinum	21.40 g/cm³
Gold	19.30 g/cm³	Potassium	0.86 g/cm³
Lead	11.30 g/cm³	Silver	10.50 g/cm³
Lithium	0.54 g/cm³	Sodium	0.97 g/cm³

$\text{Volume} = \pi R^2 h$
$= \pi (3)^2 (1.1)$
$= 9.9\pi$

$\text{density} = \frac{351.4}{9.9\pi} \approx 11.3 \text{ grams/cm}^3$

Lead