Name: ___________________________________

# Chapter 8: Matrices

## 8.2 – Matrix Operations

**OBJECTIVES:**

- Find the sum, difference, and product of two matrices
- Find the product of a scalar and a matrix

❖ Matrix Addition & Matrix Subtraction

- ➢ We add (or subtract) two matrices of the same dimension by adding (or subtracting) their corresponding entries.
- ➢ Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be matrices of order $m \times n$.
  - ▪ The sum $A + B$ is the $m \times n$ matrix $A + B = [a_{ij} + b_{ij}]$
  - ▪ The difference $A - B$ is the $m \times n$ matrix $A - B = [a_{ij} - b_{ij}]$

❖ Scalar Multiplication

  - ▪ Scalar means constant or number.
- ➢ We multiply each entry by the scalar.
- ➢ The product of the real number $k$ and the $m \times n$ matrix $A = [a_{ij}]$ is the $m \times n$ matrix $kA = [ka_{ij}]$

**EXAMPLES:**

Find the sum, difference, and or product of the matrices.

1. $\begin{bmatrix} 6 & -4 \\ 1 & 5 \\ -5 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ -6 & 2 \\ 2 & 3 \end{bmatrix}$

2. $-2\begin{bmatrix} 0 & -1 \\ 4 & -2 \\ -6 & 6 \end{bmatrix}$

3. $-5\left(\begin{bmatrix} 2 & -4 \\ -3 & -2 \end{bmatrix} - \begin{bmatrix} -4 & -1 \\ -5 & 3 \end{bmatrix}\right)$

4. $5\begin{bmatrix} -3 & -2 \\ -6 & 0 \end{bmatrix} - \begin{bmatrix} -3 & -6 \\ 1 & 0 \end{bmatrix}$

❖ Equal Matrices

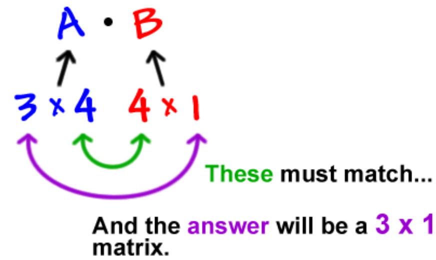
- ➢ Equal matrices have the same dimensions and equal corresponding elements.
  - ▪ You can use the definition of equal matrices to find unknown values in matrix elements.

**EXAMPLE:**

5. Find the value of each variable: $\begin{bmatrix} 2 & 4 \\ 8 & 4.5 \end{bmatrix} = \begin{bmatrix} 4x - 6 & 5 - 10t \\ 4x & 15t + 1.5x \end{bmatrix}$

- Matrix Multiplication
  - Caution!
    - Matrix multiplication can be performed only if the number of columns of the first matrix is equal to the number of rows of the second matrix.

Say the size of matrix **A** is **3 x 4**
and the size of matrix **B** is **4 x 1**
and we want **A • B**...




    - Matrix multiplication is not commutative!
      - That is, $AB \neq BA$.

**EXAMPLES:**

Determine whether the product exists.

$F = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$ $\quad G = \begin{bmatrix} -3 & 6 & 2 \\ 2 & -4 & 1 \end{bmatrix}$ $\quad H = \begin{bmatrix} -5 \\ 6 \end{bmatrix}$ $\quad J = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

6. $FG$ 7. $GF$ 8. $FH$ 9. $HG$ 10. $HJ$

- Matrix Multiplication
  - An $m \times n$ matrix $A$ and an $n \times p$ matrix $B$ can be multiplied together to form a new $m \times p$ matrix, $C$. The value of the entry in the $i$th row and $j$th column of $C$ is the product of the $i$th row of $A$ and the $j$th column of $B$.
  - Observe:

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} a_1x_1 + b_1x_2 & a_1y_1 + b_1y_2 \\ a_2x_1 + b_2x_2 & a_2y_1 + b_2y_2 \end{bmatrix}$$

**EXAMPLES:**

Find the product of the matrices, if possible.

11. $\begin{bmatrix} -3 & -6 & -1 \end{bmatrix} \cdot \begin{bmatrix} -4 & -4 \\ -3 & -6 \\ -4 & 6 \end{bmatrix}$

12. $\begin{bmatrix} 3 & 2 \\ -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 4 & 4 & 3 \\ 6 & 6 & -3 \end{bmatrix}$

# 8.3.D1 – DETERMINANTS & CRAMER'S RULE

OBJECTIVES:

- Find the value of the determinant of a matrix
- Use Cramer's Rule so solve systems of equations

❖ Determinant of a 2×2 Matrix

➢ If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$

EXAMPLES:

Find the value of each determinant.

1. $\begin{vmatrix} 24 & 6 \\ -13 & -4 \end{vmatrix}$

2. $\begin{vmatrix} -6 & 7 \\ -9 & 10 \end{vmatrix}$

❖ Cramer's Rule

➢ Determinants can be used to solve a system of linear equations.

➢ The solution to the system $\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$ is $(x, y)$ where $x = \dfrac{Dx}{D}$ & $y = \dfrac{Dy}{D}$. $D$ is the coefficient matrix: $D = \begin{vmatrix} a & b \\ d & e \end{vmatrix}$, $Dx = \begin{vmatrix} c & b \\ f & e \end{vmatrix}$, & $Dy = \begin{vmatrix} a & c \\ d & f \end{vmatrix}$

EXAMPLES:

Solve each system of equations by using Cramer's Rule.

3. $\begin{cases} 5x + 4y = -1 \\ 2x - y = 10 \end{cases}$ $D =$ $Dx =$ $Dy =$

4. $\begin{cases} 3x + 2y = 5 \\ 5x - 6y = 11 \end{cases}$ $D =$ $Dx =$ $Dy =$

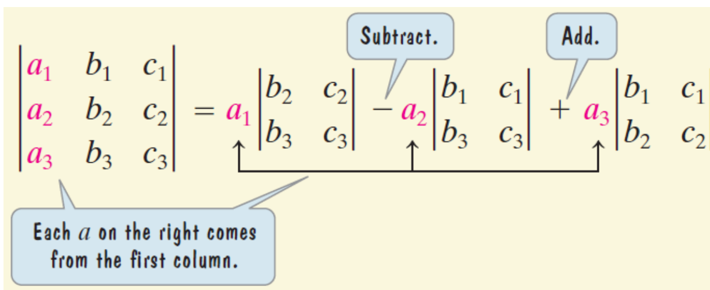
5. $\begin{cases} 4x + 7y = 22 \\ 8x - 2y = -5 \end{cases}$ $D =$ $Dx =$ $Dy =$

# 8.3.D2 – DETERMINANTS & CRAMER'S RULE

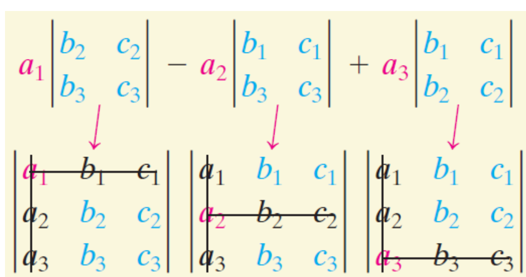
**OBJECTIVES:**

- Find the value of the determinant of a matrix
- Use Cramer's Rule so solve systems of equations

❖ Determinant of a 3×3 Matrix – Expansion by Minors




- ➢ The numerical factors – $a_1, a_2$ & $a_3$ – are elements from the first column of the third-order determinant.
  - ▪ A minus sign precedes the second term: $a_2$
- ➢ The minor – the second-order determinant – is obtained by crossing out the row and the column containing the numerical factor.



**EXAMPLE:**

Find the value of the determinant.

1. $\begin{vmatrix} 2 & 1 & 7 \\ -5 & 6 & 0 \\ -4 & 3 & 1 \end{vmatrix}$

- Cramer's Rule
  - Determinants can be used to solve a system of equations in three variables.

*If*

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

*Then*

$$x = \frac{D_x}{D}, y = \frac{D_y}{D}, \& \ z = \frac{D_z}{D}$$

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

*These are the coefficient of the variables.*

$$D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

*Replace x-coefficients in D with the constants on the right of the three equations.*

$$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

*Replace y-coefficients in D with the constants on the right of the three equations.*

$$D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

*Replace z-coefficients in D with the constants on the right of the three equations.*

## EXAMPLES:

Solve each system of equations by using Cramer's Rule.

2. $\begin{cases} x + 2y - z = -4 \\ x + 4y - 2z = -6 \\ 2x + 3y + z = 3 \end{cases}$

$D =$

$D_x =$

$D_y =$

$D_z =$

3. $\begin{cases} 3x - 2y + z = 16 \\ 2x + 3y - z = -9 \\ x + 4y + 3z = 2 \end{cases}$

$D =$

$D_x =$

$D_y =$

$D_z =$

# 8.3.D3 – INVERSE MATRICES

OBJECTIVES:

- Find inverse matrices
- Use inverse matrices to solve systems of equations

❖ Inverse Matrices
  - ➢ A matrix with an inverse is said to be <u>invertible</u>.
    - ▪ An $n \times n$ matrix $A$ is invertible if and only if $\det A \neq 0$.
  - ➢ If $\det A \neq 0$, then the inverse matrix exists and:

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

EXAMPLES:

Determine whether the matrix has an inverse. If so, find its inverse maxtrix.

1. $\begin{bmatrix} 7 & -2 \\ -9 & 2 \end{bmatrix}$

2. $\begin{bmatrix} -2 & -1 \\ -8 & -4 \end{bmatrix}$

- Solving Systems of Linear Equations Using Matrix Equations
  - Let $A$ be the coefficient matrix of a system of $n$ linear equations in $n$ variables given by $AX = B$, where $X$ is the $n \times 1$ matrix of variables and $B$ is the $n \times 1$ matrix of numbers on the right-hand side of the equations.
  - If $A^{-1}$ exists, then the system of equations has the unique solution: $X = A^{-1}B$

Solve the system:

$$\begin{gathered} 4x - 2y = 0 \\ -x + y = 10 \end{gathered}$$

Write the matrices: $A$ and $B$.

$$A = \begin{bmatrix} 4 & -2 \\ -1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

Determine if $A^{-1}$ exists.
Recall: A matrix is invertible if $ad - bc \neq 0$.

$ad - bc = 2$
$A^{-1}$ exists!

Find $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
*No need to multiply by the fraction.*

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}$$

Write & multiply: $X = A^{-1} \times B$
*Then multiply by the fraction.*

$$X = A^{-1} \times B = \frac{1}{2}\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

The system of equations has the unique solution: $X = A^{-1}B$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 20 \\ 40 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

$x = 10$ & $y = 20$

## EXAMPLES:

Write the system of equations as a matrix equation, $AX = B$, and solve using an inverse matrix.

3. $\begin{cases} 5x + 3y = -5 \\ 7x + 5y = -11 \end{cases}$

<u>Write matrix $A$.</u>

<u>Does $A^{-1}$ exist?</u>

<u>Find $A^{-1}$.</u>

<u>Write & multiply: $X = A^{-1}B$</u>

4. $\begin{cases} x + 7y = 1 \\ 2x + 5y = -7 \end{cases}$

<u>Write matrix $A$.</u>

<u>Does $A^{-1}$ exist?</u>

<u>Find $A^{-1}$.</u>

<u>Write & multiply: $X = A^{-1}B$</u>

- Using Inverses to Solve Systems of Equations w/Three Variables
  - If $AX = B$ has a unique solution, then $X = A^{-1}B$.
    - To solve a system of equations, multiply $A^{-1}$ & $B$ to find $X$.

Solve the system:

$$\begin{cases} x - y + z = 2 \\ -2y + z = 2 \\ -2x - 3y = 0.5 \end{cases}$$

Write the matrices: $A$ and $B$.

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{bmatrix} \ \& \ B = \begin{bmatrix} 2 \\ 2 \\ 0.5 \end{bmatrix}$$

Given: $A^{-1}$

$$A^{-1} = \begin{bmatrix} 3 & -3 & 1 \\ -2 & 2 & -1 \\ -4 & 5 & -2 \end{bmatrix}$$

Multiply: $X = A^{-1} \times B$

$$X = A^{-1} \times B = \begin{bmatrix} 3 & -3 & 1 \\ -2 & 2 & -1 \\ -4 & 5 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 0.5 \end{bmatrix}$$

The system of equations has the unique solution: $X = A^{-1}B$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.5 \\ 1 \end{bmatrix}$$

**EXAMPLE:**

Write the system of equations as a matrix equation, $AX = B$, and solve using the inverse matrix.

5. $\begin{cases} x - 6y + 3z = 11 \\ 2x - 7y + 3z = 14 \\ 4x - 12y + 5z = 25 \end{cases}$ $\quad A^{-1} = \begin{bmatrix} 1 & -6 & 3 \\ 2 & -7 & 3 \\ 4 & -12 & 5 \end{bmatrix}$