Name: ______________________________

Chapter 8/9 – Trigonometric Equations & Identities

8.4.D1 – Trigonometric Equations (Part 1)

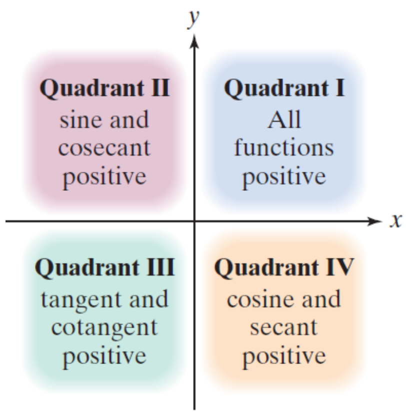
Objective:

- Solve trigonometric equations, with solutions in degrees, using the unit circle and reference angles

- Trigonometric Equations
  - A trigonometric equation is an equation that contains a trigonometric expression with a variable.
    - The values that satisfy the equation are its solution.
- Know the Signs

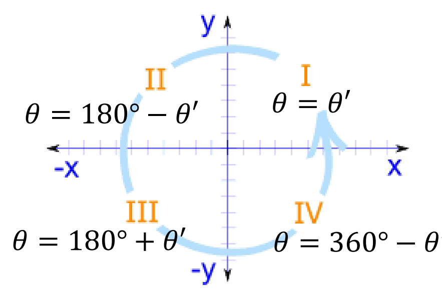
WHAT IS NEGATIVE IN Q2?

WHAT IS NEGATIVE IN Q3?



WHAT IS NEGATIVE IN Q4?

- Using a Calculator to Find an Approximate Value of θ (in Degrees)
  - Caution! The calculator can give the "wrong" value of θ
  - Recall: $\tan^{-1} x = \theta$ & $\sin^{-1} x = \theta$
    - If $x > 0$, then the calculator will provide an angle in quadrant I, such that $0° < \theta < 90°$
    - If $x < 0$, then the calculator will provide an angle in quadrant IV, such that $-90° < \theta < 0°$
  - Use the calculator's value of θ as a reference angle.
    - Ignore the negative.



  - Recall: $\cos^{-1} x = \theta$
    - If $x > 0$, then the calculator will provide an angle in quadrant I, such that $0° < \theta < 90°$
    - If $x < 0$, then the calculator will provide an angle in quadrant II, such that $90° < \theta < 180°$
      - Subtract the angle from 180° to obtain the reference angle.

- Finding Angle Measures in Degrees
  - If you know the value of a trig function of an angle, a calculator will give you the measure of ***ONE*** angle with that value.
    - Use $\sin^{-1} x$, $\cos^{-1} x$, or $\tan^{-1} x$
    - Use your knowledge of the signs of various trig functions in each quadrant to decide where there is a second angle that has the same value.
    - Use your knowledge of reference angles to determine the second angle's measure.

Find the measure of each angle θ, where $0° \leq \theta < 360°$, to the nearest tenth of one degree.

1) $\sin\theta = -0.5864$

a. Ask yourself: Where is sine negative?

Sine is negative in Quadrants 3 & 4.

b. Find θ^R : Calculate $\sin^{-1}(-0.5864)$

$\theta^R = \sin^{-1}(-0.5864) \approx -35.9$

Ignore the negative; this is your REFERENCE ANGLE

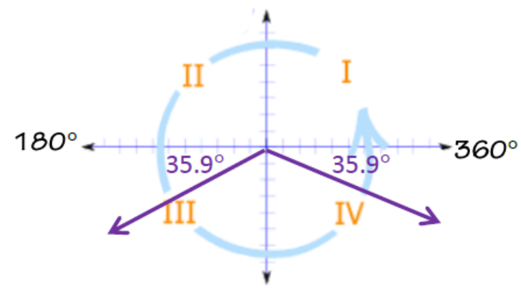
c. Find θ

In Quadrant 3

$\theta = 180 + 35.9 = 215.9°$

In Quadrant 4

$\theta = 360 - 35.9 = 324.1°$



❖ Calculating Inverses of the Reciprocal Trig Functions

Inverse Cosecant: $\sin^{-1}\left(\frac{1}{value}\right)$

Inverse Secant: $\cos^{-1}\left(\frac{1}{value}\right)$

Inverse Cotangent: $\tan^{-1}\left(\frac{1}{value}\right)$

2) $\sec\theta = 3.1909$

a. Ask yourself: Since secant is the reciprocal of cosine, where is cosine positive?

Cosine is positive in Quadrants 1 & 4.

b. Find θ^R : Calculate $\cos^{-1}(1/3.1909)$

$\theta^R = \cos^{-1}(1/3.1909) \approx 37.2$

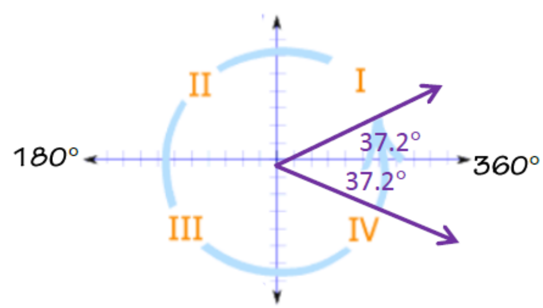
c. Find θ

In Quadrant 1

$\theta = 37.2°$

In Quadrant 4

$\theta = 360 - 37.2 = 322.8°$



EXAMPLES: Solve; finding all solutions where $0° \leq \theta \leq 360°$, to the nearest tenth of one degree. Make sure the calculator is in DEGREE mode.

1. $\cos\theta = -0.3623$

2. $\tan\theta = 2.544$

3. $\csc\theta = -1.253$

4. $\cot\theta = -3.265$

- Equations Involving a Single Trigonometric Function
  - Isolate the function on one side of the equation.
  - Solve for the variable.

EXAMPLES: Solve, finding all solutions where $0° \leq \theta \leq 360°$, to the nearest tenth of one degree. Make sure the calculator is in DEGREE mode.

5. $2\cos\theta - 1 = -1.2814$

6. $3\sin\theta + 3 = 5\sin\theta + 2$

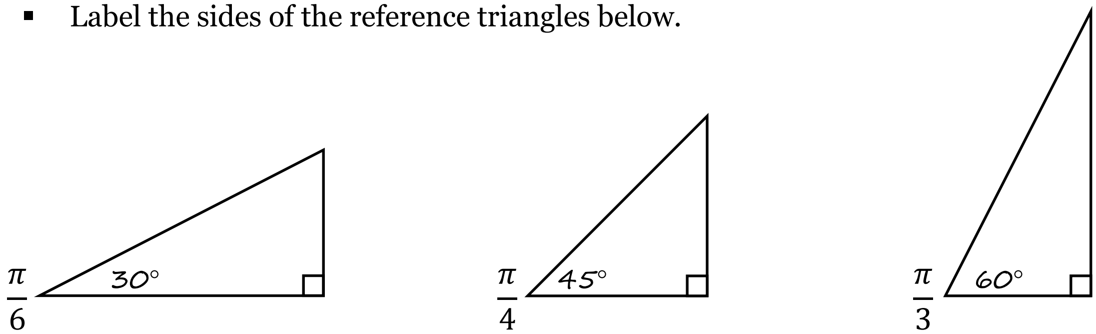
8.4.D2 – TRIGONOMETRIC EQUATIONS (PART 1)

Objective:

- Solve trigonometric equations, with solutions in radians, using the unit circle and reference angles
- Find ALL solutions of a trigonometric equation

❖ Finding EXACT Solutions (in Radians) of a Trigonometric Equation

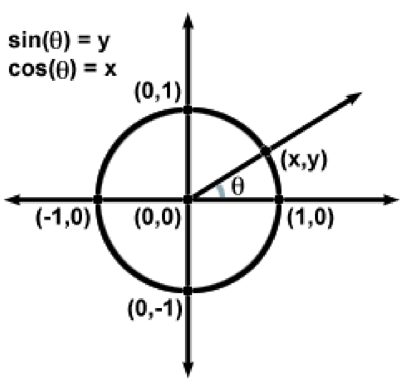
- ➢ Know: Trig Functions of Special Angles
  - Label the sides of the reference triangles below.



  - Find the exact values – in simplest form – of sine, cosine, and tangent for a 30°, 45° and 60° reference angle.

	30°	45°	60°
SINE			
COSINE			
TANGENT			

- ➢ Know: Trig Functions of Quadrantal Angles
  - Use the definitions sine, cosine, and tangent, the unit circle below, and the fact that r = 1 to find the exact values of sine, cosine, and tangent for quadrantal angles.



		SINE	**COSINE**	**TANGENT**
0°, 360°	$0, 2\pi$			
90°	$\frac{\pi}{2}$			
180°	π			
270°	$\frac{3\pi}{2}$			

❖ Equations Involving a Single Trigonometric Function

- ➢ Isolate the function on one side of the equation.
- ➢ Solve for the variable.

EXAMPLES: Solve, finding all solutions in the interval $[0, 2\pi]$. Use a reference triangle to assist you.

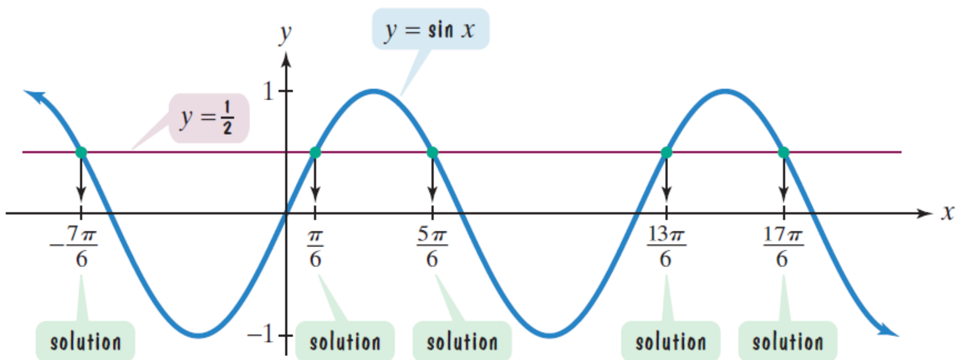
1. $-2\sin x = \sqrt{2}$
2. $\cos x = 0$
3. $\tan x = -1$

- Finding ALL Solutions of a Trigonometric Equation

$$\sin x = \frac{1}{2} \quad \rightarrow \quad x = \frac{\pi}{6}$$

Is that the only solution?

- Recall that the domain of sine is *all real numbers*.
  - There are technically infinitely many solutions.



- Representing ALL Solutions
  - The period of the sine function is 2π, first find all solutions in $[0, 2\pi)$.

$$x = \frac{\pi}{6} \text{ or } x = \frac{5\pi}{6}$$

The period of tangent is π.

  - Any multiple of 2π can be added to these values and the sine is still ½. Thus, all solutions of the equation are given by

$$x = \frac{\pi}{6} + 2n\pi \text{ or } x = \frac{5\pi}{6} + 2n\pi$$

where n is any integer.

  - By choosing any two integers, such as $n = 0$ and $n = 1$, we can find *some* solutions of the equation.

Let $n = 0$.

$$x = \frac{\pi}{6} + 2 \cdot 0\pi = \frac{\pi}{6} \qquad x = \frac{5\pi}{6} + 2 \cdot 0\pi = \frac{5\pi}{6}$$

Let $n = 1$.

$$x = \frac{\pi}{6} + 2 \cdot 1\pi = \frac{\pi}{6} + 2\pi = \frac{\pi}{6} + \frac{12\pi}{6} = \frac{13\pi}{6}$$

$$x = \frac{5\pi}{6} + 2 \cdot 1\pi \qquad x = \frac{5\pi}{6} + 2\pi \qquad = \frac{5\pi}{6} + \frac{12\pi}{6} = \frac{17\pi}{6}$$

EXAMPLES: Solve the equation. Find all solutions. Exact values, no decimals.

4. $3\sin x - 2 = 5\sin x - 1$

5. $5\cos x = 3\cos x + \sqrt{3}$

- Trigonometric Equations Quadratic in Form
  - Quadratic Form: $au^2 + bu + c = 0$, where u is a trigonometric function and $a \neq 0$.

$2\cos^2 x + \cos x - 1 = 0$ — The form of this equation is $2u^2 + u - 1 = 0$ with $u = \cos x$.

$2\sin^2 x - 3\sin x + 1 = 0$ — The form of this equation is $2u^2 - 3u + 1 = 0$ with $u = \sin x$.

- Solving Methods
  - Factor & use the Zero Product Property
  - Quadratic Formula
  - Square Root Property
    - If $u^2 = c$, then $u = \pm\sqrt{c}$.

EXAMPLES: Solve, finding all solutions in the interval $[0, 2\pi]$.

6. $2\cos^2 x + \cos x - 1 = 0$

7. $4\sin^2 x - 1 = 0$

9.1.D1 – Simplifying Trigonometric Expressions

Objectives:

- Use the fundamental identities to simplify trigonometric expressions and solve equations

- Fundamental Identities
  - Trigonometric identities are equations that are true for all real numbers for which the trigonometric expressions in the equations are defined.

- Basic Trigonometric Identities
  - Reciprocal Trigonometric Functions

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

  - Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

  - Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \cot^2\theta = \csc^2\theta \qquad 1 + \tan^2\theta = \sec^2\theta$$

  - Cofunction Identities

$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right) \qquad \tan\theta = \cot\left(\frac{\pi}{2} - \theta\right) \qquad \sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$$

$$\cos\theta = \sin(90° - \theta) \qquad \cot\theta = \tan(90° - \theta) \qquad \csc\theta = \sec(90° - \theta)$$

  - Odd-Even Identities

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x \qquad \tan(-x) = -\tan x$$

$$\csc(-x) = -\csc x \qquad \sec(-x) = \sec x \qquad \cot(-x) = -\cot x$$

EXAMPLES: Rewrite each expression by changing to sines and cosines. Then simplify the resulting expression.

1. $\sec\alpha \cot\alpha$
2. $\sec\theta - \tan(-\theta)$
3. $\dfrac{\sec\beta}{\cos\beta}$

The skills used to manipulate and simplify algebraic expressions can also be used with trigonometric expressions.

EXAMPLES: Use the fundamental identities and algebra to simplify the expression to either a constant or a basic trigonometric function.

4. $\sin^3 x + \cos^2 x \sin x$

5. $\dfrac{\cos x}{1 - \sin x} - \dfrac{\sin x}{\cos x}$

- Writing Equivalent Trigonometric Expressions
  - The patterns for factoring polynomials can be used along with the fundamental identities to convert one trigonometric expression to another.
    - $a^2 - b^2 = (a + b)(a - b)$
    - $a^2 + 2ab + b^2 = (a + b)^2$
    - $a^2 - 2ab + b^2 = (a - b)^2$

EXAMPLES: Find all solutions to the equation in the interval $[0, 2\pi)$ without a calculator.

6. $\cos x - 2\sin^2 x + 1 = 0$

7. $4\sin^2 x - \dfrac{4}{\csc x} + \tan x \cot x = 0$

9.1.D2 – Verifying Trigonometric Identities

Objective:

- Use the fundamental identities to verify trigonometric identities

❖ Using Fundamental Identities to Verify Other Identities

- ➢ To ***verify an identity***, we show that one side of the identity can be simplified so that it is identical to the other side.
  - Each side of the equation is manipulated ***independently*** of the other side of the equation.

❖ Verifying an Identity: Strategies

- ➢ Write everything in ***sine*** & ***cosine***.
- ➢ Keep your eye on the goal.
  - Work with ***one side only***; continuously refer back to the other side to see what you are trying to obtain.
- ➢ When one side contains only ***one*** trig function, attempt to rewrite all the functions on the other side in terms of that function.
- ➢ Use the ***Pythagorean identities*** to substitute for expressions equal to ***1***.
- ➢ Analyze the identity and look for opportunities to apply the fundamental identities.
- ➢ Perform algebraic operations, if necessary.
  - Factor
  - Simplify complex rational expressions.
    - Multiply all terms by the common denominator
  - Find the LCD and combine fractions.
  - Separate a single term quotient into two terms:

$$\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c} \text{ and } \frac{a-b}{c}=\frac{a}{c}-\frac{b}{c}$$

  - Combine like terms.
  - Multiply both numerator & denominator by the same expression to obtain an equivalent fraction.
  - Work with conjugates.
    - Conjugates are easy to find; all you do is change the sign.

Common Misconception

You cannot perform operations to the quantities from each side of an unverified identity as you do with equations. Until an identity is verified it is not considered an equation, so the properties of equality do not apply.

❖ Proving/verifying an identity is ***not*** the same as solving an equation.

- ➢ Equation-solving techniques, such as operating on both sides ***are not to be used*** in the proof of an identity.
  - You do not verify an identity by adding, subtracting, multiplying, or dividing each side by the same expression. If you do this, you have already assumed that the given statement is true.
    - You do not know that it is true until after you have verified it.

EXAMPLES: Prove/verify each identity.

1. $\sec\alpha\cot\alpha = \csc\alpha$ *Write everything in sine & cosine.*

2. $\cos\beta - \cos\beta\sin^2\beta = \cos^3\beta$ *Factor.*

3. $\dfrac{1-\cos\delta}{\sin\delta} = \csc\delta - \cot\delta$ *Separate a single term quotient.*

4. $\dfrac{\cos\mu}{1+\sin\mu} + \dfrac{1+\sin\mu}{\cos\mu} = 2\sec\mu$ *Find the LCD and combine fractions.*

5. $\dfrac{\sin\theta}{1+\cos\theta} = \dfrac{1-\cos\theta}{\sin\theta}$

Work with conjugates.

6. $\dfrac{1}{1+\cos\omega} + \dfrac{1}{1-\cos\omega} = 2 + 2\cot^2\omega$

Work with both sides, SEPARATELY.

7. $\dfrac{\csc^2 x - 1}{\csc^2 x} = \cos^2 x$

I see Pythagorean Identities!

9.2.D1 – Sum & Difference Formulas

Objectives:

- Use the sum and difference formulas for sine, cosine, and tangent to find the exact value of trigonometric functions
- Use the sum and difference identities to verify trigonometric identities

❖ Sine – Sum & Difference Formulas

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$ $\qquad$ $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

❖ Cosine – Sum & Difference Formulas

$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$ $\qquad$ $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

EXAMPLES: Use a sum or difference identity to find the exact value of each expression.

1. $\sin 15°$
2. $\cos\left(\frac{2\pi}{3}+\frac{\pi}{4}\right)$

EXAMPLES: Use a sum or difference identity to find the exact value of each expression.

First draw two reference triangles: one for α and one for β.

$\sin\alpha=\frac{4}{5}$ $\qquad$ $\tan\beta=-\frac{7}{24}$

$0<\alpha<\frac{\pi}{2}$ $\qquad$ $\frac{3\pi}{2}<\beta<2\pi$

3. $\sin(\alpha+\beta)$
4. $\cos(\alpha+\beta)$

EXAMPLES: Verify each identity.

5. Prove the cofunction identity:

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

6. $\dfrac{\cos(\alpha - \beta)}{\cos\alpha \sin\beta} = \tan\alpha + \cot\beta$

7. $\sin(2\theta) = 2\sin\theta\cos\theta$

Hint: Use $\sin(2\theta) = \sin(\theta + \theta)$.

9.1.D3 – Double-Angle Identities

Objectives:

- Derive the double-angle identities for sine, cosine, and tangent
- Use the double-angle identities to find exact values and to verify trigonometric identities

❖ Deriving the Double-Angle Identities

- Use the Sum Identity for Sine to derive $\sin 2\alpha$

  See Example 7 from 9.2.D1.

- Use the Sum Identity for Cosine to derive $\cos 2\alpha$

  $\cos 2\alpha = \cos(\alpha + \alpha)$

❖ Double-Angle Identities

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$\cos 2\alpha = 2\cos^2\alpha - 1$$

$$\cos 2\alpha = 1 - 2\sin^2\alpha$$

EXAMPLES: Use a Double-Angle Identity to find the exact value for each trigonometric function.

First draw a reference triangle for α:

$\sin\alpha = \frac{12}{13}$ & $90° < \alpha < 180°$

1. $\sin 2\alpha$

2. $\cos 2\alpha$

3. $\tan 2\alpha$

EXAMPLES: Simplify the expression.

4. $\dfrac{\sin 2\beta}{\cos \beta}$

5. $\dfrac{\cos 2\alpha}{\cos \alpha + \sin \alpha}$

EXAMPLES: Verify each identity.

6. $\cos 2\theta = \dfrac{1 - \tan^2\theta}{\sec^2\theta}$

7. $\cos 3\alpha = 4\cos^3\alpha - 3\cos \alpha$

8. $(\sin \beta - \cos \beta)^2 = 1 - \sin 2\beta$

8.4.D3 – Trigonometric Equations (Part 2)

Objective:

- Find the exact solutions (in radians) of trigonometric equations using identities
- Find the approximate solutions (in degrees) of trigonometric equations using identities

❖ Solving Trigonometric Equations

- Use the rules of Algebra.
- When an equation contains two or more trig functions, it is helpful to express the terms using one function only.
- Trigonometric equations quadratic in form can be expressed as $au^2 + bu + c = 0$, where u is a trigonometric function and $a \neq 0$.
  - Solving Methods
    - Factor & use the Zero Product Property
    - Square Root Property: If $u^2 = c$, then $u = \pm\sqrt{c}$.
- Factoring can be used to separate two different trigonometric functions in an equation.
- Identities are used to solve some trigonometric equations.

Examples: Find the exact solutions to each equation, where $0 \leq x \leq 2\pi$.

1. $\cos 4x = -\frac{\sqrt{3}}{2}$
2. $\tan 3x = 1$
3. $\cos 2x + 3\sin x - 2 = 0$
4. $\sin x \cos x = \frac{1}{2}$