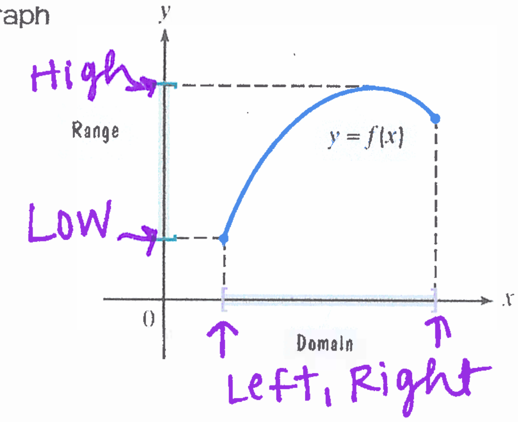
 Name: ____________________

Chapter 3: Functions & their transformations

3.APK.1 – DOMAIN & RANGE

OBJECTIVE: Determine the domain and range of a function given its graph



❖ FUNCTIONS, DOMAIN, & RANGE

➢ A function from a set D to a set R is a rule that assigns to element in D a unique element of R

- Domain—the set D of all inputs; the variable x
- Range—the set R of all outputs, the variable y

❖ FINDING THE DOMAIN & RANGE OF A FUNCTION

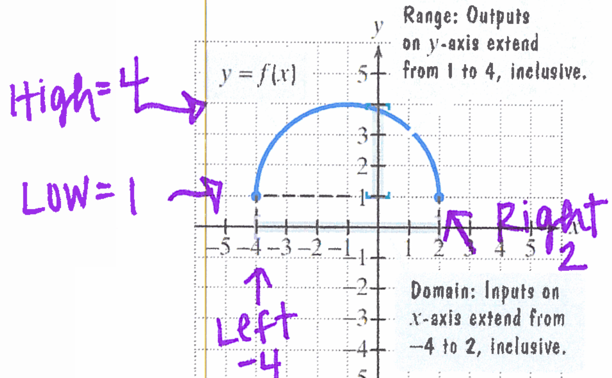


FIGURE 1.25

DOMAIN:

Using Set-Builder Notation

$\{ x \mid -4 \le x \le 2 \}$

The set of all x such that x is greater than or equal to −4 and less than or equal to 2.

Using Interval Notation

Left→ $[-4, 2]$ ←Right

The square brackets indicate −4 and 2 are included. Note the square brackets on the x-axis in **Figure 1.25**.

RANGE:

$\{ y \mid 1 \le y \le 4 \}$

The set of all y such that y is greater than or equal to 1 and less than or equal to 4.

Low→ $[1, 4]$ ←High

The square brackets indicate 1 and 4 are included. Note the square brackets on the y-axis in **Figure 1.25**.

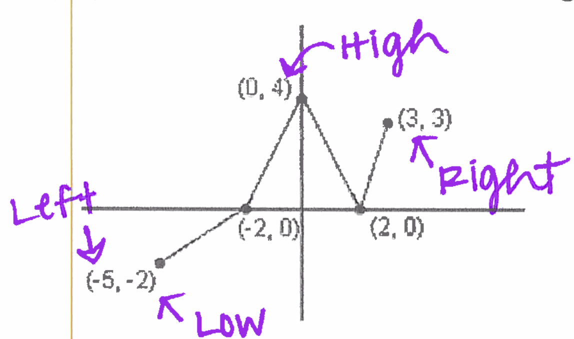
❖ REPRESENTING W/ INTERVAL NOTATION

$(-\infty, \_)$	$(\_, \infty)$	[Brackets]	(Parentheses)
Left forever or low forever	Right forever up forever	solid on graph	open circles or asymptotes

EXAMPLES: FINDING THE DOMAIN & RANGE OF A FUNCTION GRAPHICALLY

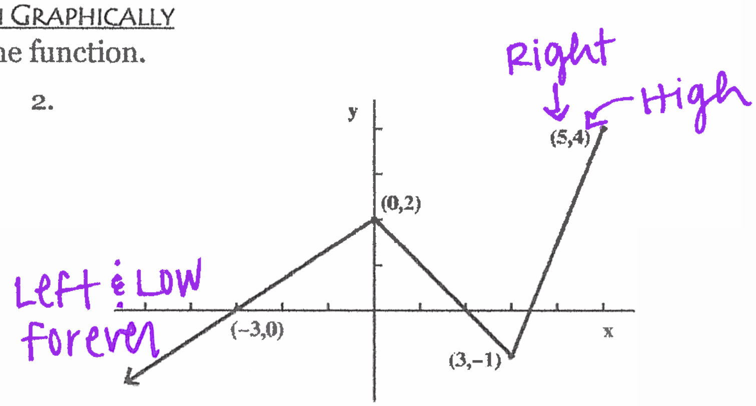
Use the graph to determine the domain and range of the function.

1.



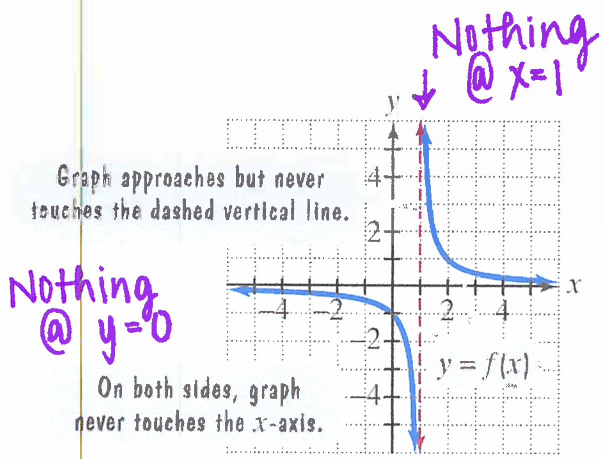
DOMAIN	RANGE
$[-5, 3]$	$[-2, 4]$

2.

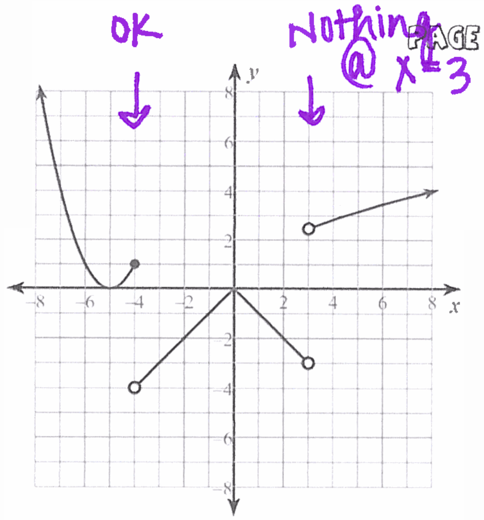


DOMAIN	RANGE
$(-\infty, 5]$	$(-\infty, 4]$

3.



4.



DOMAIN	RANGE
$(-\infty,1) \cup (1,\infty)$	$(-\infty,0) \cup (0,\infty)$

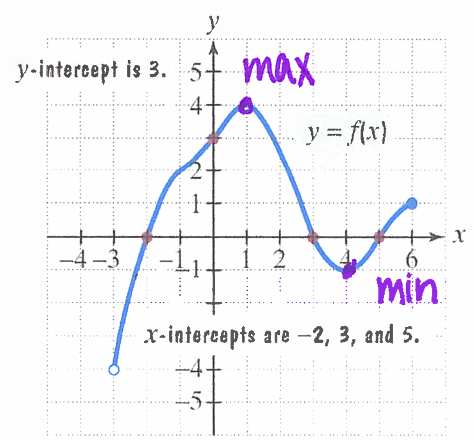
DOMAIN	RANGE
$(-\infty,3) \cup (3,\infty)$	$(-4,\infty)$

3.APK.2 – CHARACTERISTICS OF FUNCTIONS

<u>OBJECTIVE:</u> Analyze the graph of a function for: domain, range, x-intercept(s), y-intercept; intervals on which a function is increasing, decreasing, or constant; maximum & minimum values; and end behavior

❖ CHARACTERISTICS OF FUNCTIONS

- ➢ **x-intercept(s)** – where the graph crosses/touches the x-axis
  - ▪ Value at $y = 0$ or $f(x) = 0$
- ➢ **y-intercept** – where the graph crosses the y-axis
  - ▪ Value of the function at $x = 0$
- ➢ **Increasing/Decreasing/Constant Intervals**
  - ▪ Use ONLY the x-values of the function to describe the interval
    - x-values represent location while y-values represent values of the function
    - (x start, x end)
  - ▪ Use parentheses; never use brackets.
    - If we use brackets – and not parentheses – then we are saying that the value of the function is both increasing and decreasing at the same time at the same location.
- ➢ **Local Extrema: Maxima & Minima**
  - ▪ Occur ONLY at points at which a function <u>changes</u> its increasing or decreasing behavior.

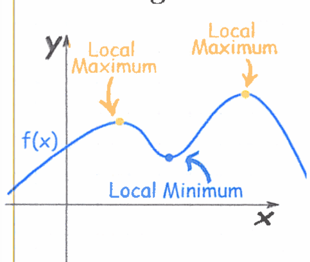


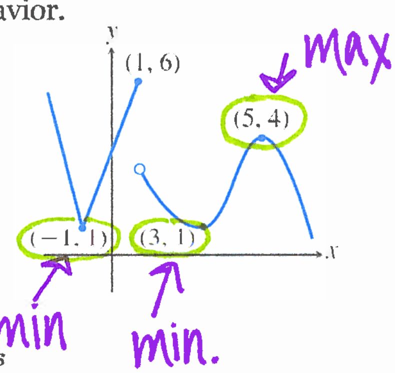
INCREASING INTERVALS: $(-3,1) \cup (4,6)$

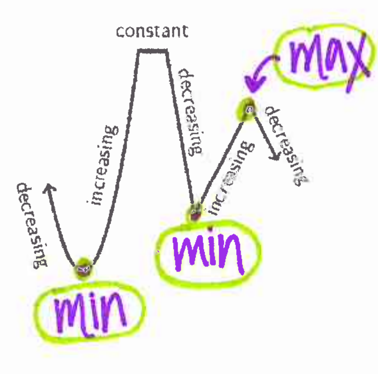
DECREASING INTERVAL: $(1,4)$

MAX OF 4 AT X = 1

MIN OF −1 AT X = 4

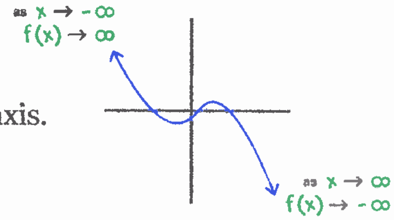


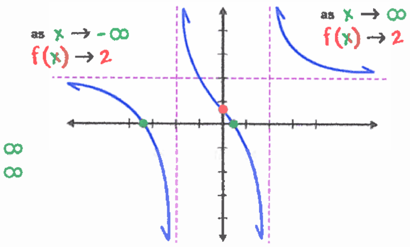




- **End behavior**

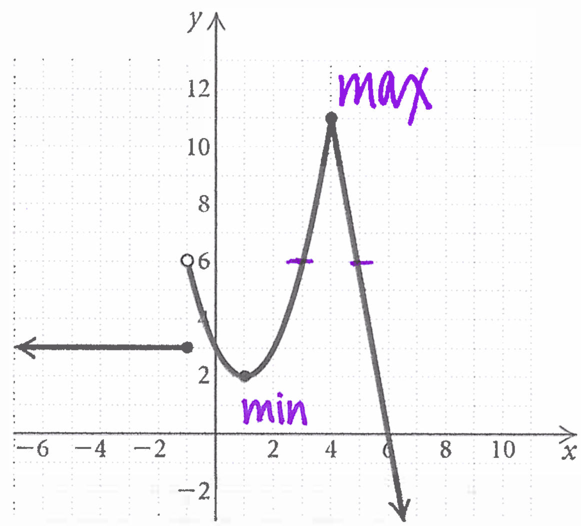
  The end behavior of a function describes the behavior of the graph at the "ends" of the x-axis.
  - Left: $\lim_{x\to-\infty} f(x) =$ a y value
  - Right: $\lim_{x\to\infty} f(x) =$ a y value





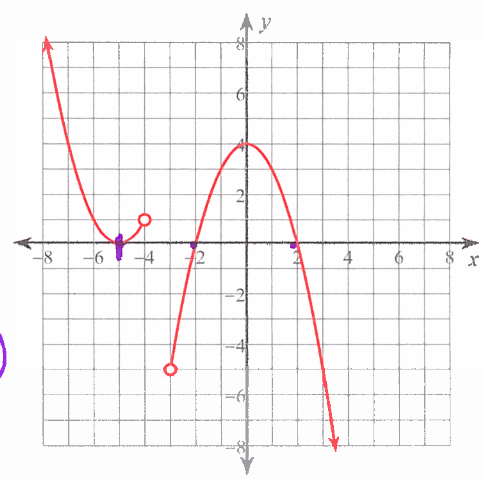
EXAMPLES: ANALYZING FUNCTIONS

Use the graph of $f(x)$ to find the following.



1. The domain: $(-\infty, \infty)$
2. The range: $(-\infty, 11]$
3. The x-intercept: 6
4. The y-intercept: 3
5. Increasing interval(s): $(1,4)$
6. Decreasing interval(s): $(-1,1) \cup (4,\infty)$
7. Constant interval(s): $(-\infty,-1)$
8. Maximum value = 11; location: $x =$ 4
9. Minimum value = 2; location: $x =$ 1
10. $f(-10) = 3$
11. Value(s) for which $f(x) = 6$ $x = 3$ & 5
12. End behavior: $\lim_{x\to-\infty} f(x) = 3$
13. End behavior: $\lim_{x\to\infty} f(x) = -\infty$

Use the graph of $f(x)$ to find the following.



14. The domain $(-\infty,-4) \cup (-3,\infty)$
15. The range $(-\infty,\infty)$
16. The x-intercept(s) $-5, -2$ & 2
17. The y-intercept 4
18. Interval(s) on which f is increasing $(-5,-4) \cup (-3,0)$
19. Interval(s) on which f is decreasing $(-\infty,-5) \cup (0,\infty)$
20. End behavior: $\lim_{x\to-\infty} f(x) = \infty$
21. End behavior: $\lim_{x\to\infty} f(x) = -\infty$

3.1 – VERTICAL & HORIZONTALS SHIFTS

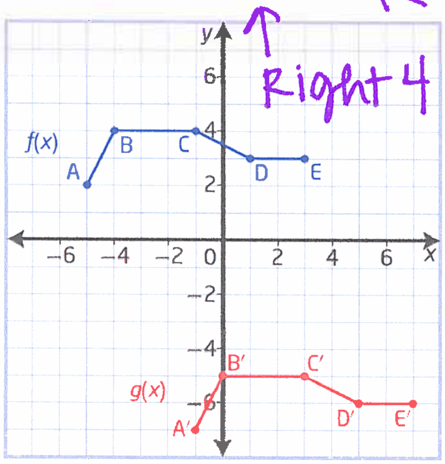
OBJECTIVES:

- Identify the effect on the graph of a function replacing $f(x)$ by $f(x) + k$ and $f(x + k)$ for specific values of k (both positive or negative)
- Describe, write a formula, graph and interpret a function that has been shifted vertically and/or horizontally

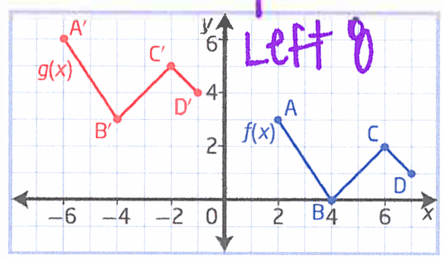
❖ EXPLORING TRANSLATIONS – *What do you notice? What do you wonder?*

Also identify the domain and range of $f(x)$, as well as the domain and range of the transformed graphs.

$g(x) = f(x - 4) - 9$ Down 9



$g(x) = f(x + 8) + 3$ up 3



	DOMAIN	RANGE
$f(x)$	[-5,3]	[2,4]
$g(x)$	[-1,7]	[-7,-5]

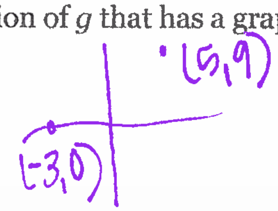
	DOMAIN	RANGE
$f(x)$	[2,7]	[0,3]
$g(x)$	[-6,-1]	[3,6]

❖ TRANSLATIONS (shifts)

FUNCTION NOTATION	DESCRIPTION	COORDINATE RULE	DOMAIN OR RANGE CHANGE?
$y = f(x - h)$	Right h	(x+h, y)	domain
$y = f(x + h)$	Left h	(x-h, y)	domain
$y = f(x) + k$	up k	(x, y+k)	Range
$y = f(x) - k$	Down k	(x, y-k)	Range

Examples:

1. The graph of $g(x)$ contains the point $(-3, 0)$. Describe the translation and then write a formula for a translation of g that has a graph containing the point $(5, 9)$.

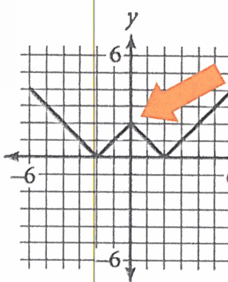


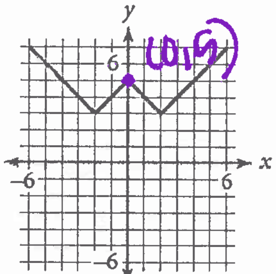
x = -3 → 5 Right 8

y = 0 → 9 up 9

g(x-8) + 9

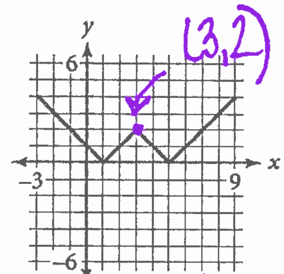
2. The graph of $y = f(x)$ is shown below. Write an equation for each related graph showing how the function has been translated.





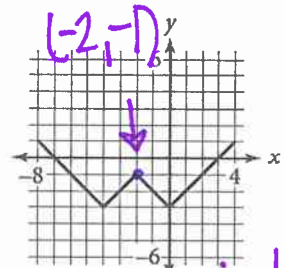
(0,5)

up 3
f(x)+3



(3,2)

Right 3
f(x-3)



(-2,-1)

Left 2 & down 3
f(x+2)-3

3. Suppose that the x-intercepts of the graph of $f(x)$ are -5 & 3. What are the x-intercepts of the graph of $y = f(x+2)$?

Left 2 = x-2

-5 → -7 ← x-ints
3 → 1

4. The domain of a function $h(x)$ is $[0, 12]$ and its range is $[-4, 2]$. What is the domain and range of $h(x+5) - 12$?

Down 12 = y-12

Left 5 = x-5

R = [-16, -10]

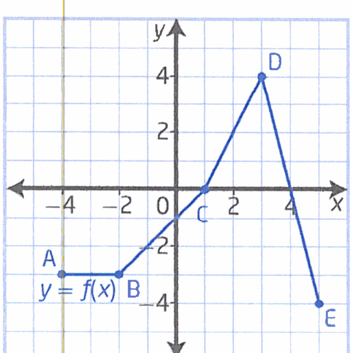
D = [-5, 7]

3.2 – VERTICAL & HORIZONTALS REFLECTIONS

OBJECTIVES:

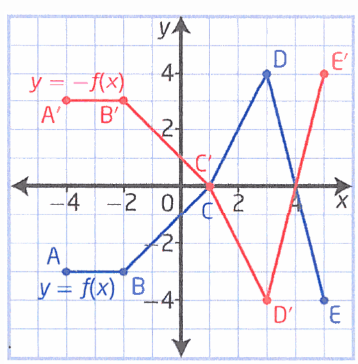
- Identify the effect on the graph of a function replacing $f(x)$ by $-f(x)$ and $f(-x)$
- Describe, write a formula, graph and interpret a function that has been reflected vertically and/or horizontally

❖ EXPLORING REFLECTIONS – *What do you notice? What do you wonder?*



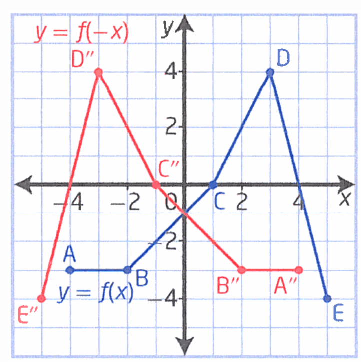
Graph $y = -f(x)$

REFLECT OVER THE X-AXIS



Graph $y = f(-x)$

REFLECT OVER THE Y-AXIS



COORDINATES OF A	COORDINATES OF E	COORDINATES OF A'	COORDINATES OF E'	COORDINATES OF A'	COORDINATES OF E'
A(-4,-3)	E(5,-4)	A'(-4,3)	E'(5,4)	A'(4,-3)	E'(-5,-4)

same opp. sign same

opposite sign same same

(same x, opposite y) (opposite x, same y)

❖ REFLECTIONS ACROSS AXES (flips)

FUNCTION NOTATION	DESCRIPTION	COORDINATE RULE	DOMAIN OR RANGE CHANGE?
$y = -f(x)$	Reflect over x-axis	$(x, -1 \cdot y)$	Range
$y = f(-x)$	Reflect over y-axis	$(-1 \cdot x, y)$	Domain

Examples:

1. The graph of $f(x)$ contains the point $(2, -3)$. What point must lie on the reflected graph if the graph is...
   a. reflected about the x-axis? $(2, 3)$
   b. reflected about the y-axis? $(-2, -3)$

2. The domain of a function $h(x)$ is $[0, 12]$ and its range is $[-4, 2]$.
   What is the domain and range of $-h(x-4)+5$?

Reflect over x-axis $-1 \cdot y$

Right 4 $x+4$

up 5 $= y+5$

D	R
$x+4$	$-1 \cdot y + 5$
$[4, 16]$	$[9, 3]$
	$[3, 9]$

❖ ORDER IS IMPORTANT!

1 ⇨	2 ⇨	3 ⇨	4 ⇨	5
REFLECTION ABOUT Y-AXIS	HORIZONTAL TRANSLATION	VERTICAL STRETCH/ compression	REFLECTION ABOUT X-AXIS	VERTICAL TRANSLATIONS

3. The graph of the parent function $f(x)$ is given. Match the transformed function with its graph.

$y = f(-x)$ B

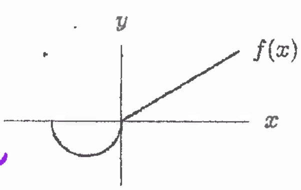
$y = -f(x)$ C

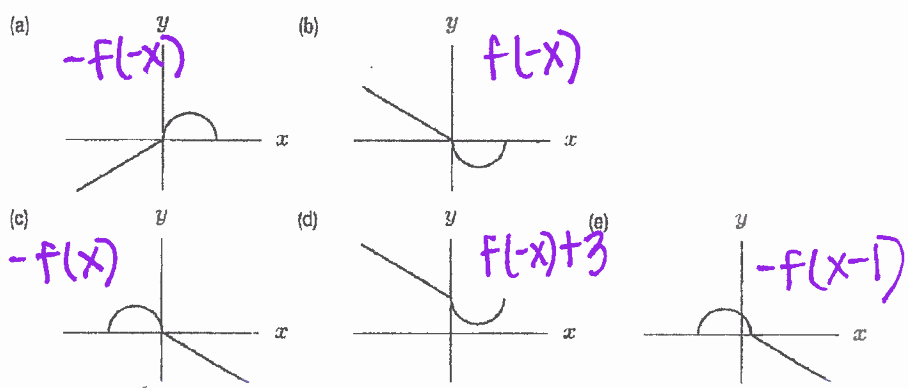
$y = f(-x) + 3$ d

$y = -f(x-1)$ E

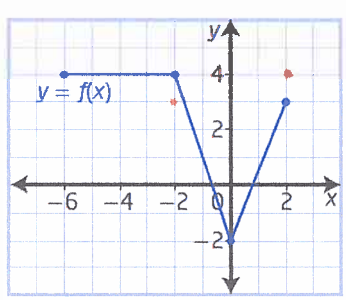
$y = -f(-x)$ A

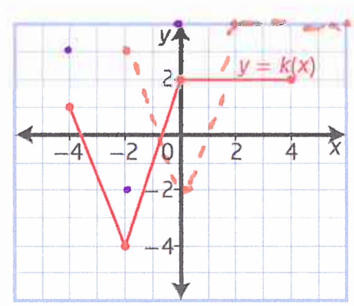
PARENT FUNCTION





4. The graph of $y = f(x)$ is shown at left. Describe the transformation and then write the equation of $k(x)$ in terms of $f(x)$.





Reflect over y-axis $= -x$

Left 2 $= f(x+2)$

Down 2 $= f(x) - 2$

$f(-1(x-2)) - 2$

$f(-x+2) - 2$

3.3 – VERTICAL STRETCHES & COMPRESSIONS

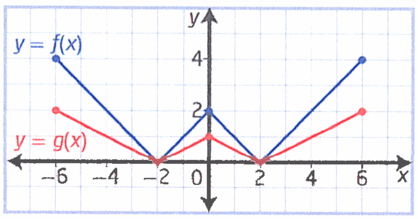
OBJECTIVES:

- Identify the effect on the graph of a function replacing $f(x)$ by $kf(x)$ for specific values of k
- Describe, write a formula, graph and interpret a function that has been reflected vertically and/or horizontally

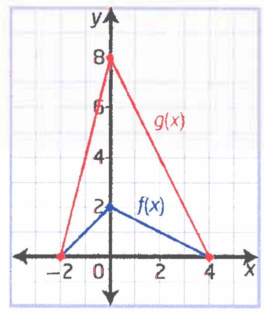
❖ EXPLORING VERTICAL SIZE CHANGES – *What do you notice? What do you wonder?*

Also identify the domain and range of $f(x)$, as well as the domain and range of the transformed graphs.

$g(x) = \frac{1}{2}f(x)$



$g(x) = 4f(x)$



same domain

½ of the original

4xs OG values

$f(x)$		$g(x)$		$f(x)$		$g(x)$	
DOMAIN	RANGE	DOMAIN	RANGE	DOMAIN	RANGE	DOMAIN	RANGE
[-6, 6]	[0, 4]	[-6, 6]	[0, 2]	[-2, 4]	[0, 2]	[-2, 4]	[0, 8]

❖ VERTICAL STRETCHES & COMPRESSIONS

FUNCTION NOTATION	DESCRIPTION	COORDINATE RULE	DOMAIN OR RANGE CHANGE?
$y = Af(x), \|A\| > 1$	vert. stretch x A	(x, A·y)	Range
$y = Af(x), 0 < \|A\| < 1$	vert. comp. x A	(x, A·y)	

Examples:

1. The graph of $f(x)$ contains the point $(3,-2)$. What corresponding point is on the graph of $g(x) = 3f(x-8)$?

   +8 x3

   (11, -6)

   stretch 3·y

   Right 8 x+8

2. The graph of $h(x)$ is found by vertically stretching the graph of $f(x)$ by a factor of 7, reflecting it about the x-axis, and then vertically shifting it down 3 units. Find a formula for $h(x)$ in terms of $f(x)$.

   $h(x) = -7f(x) - 3$

3. The function $g(x)$ is obtained from $f(x)$ by a single transformation. Use the tables below to find a formula for $g(x)$ in terms of $f(x)$.

x	-4	-2	0	2	4
$f(x)$	12	-4	-2	4	6

x	-4	-2	0	2	4
$g(x)$	36	−12	−6	12	18

x3

vert. stretch x3

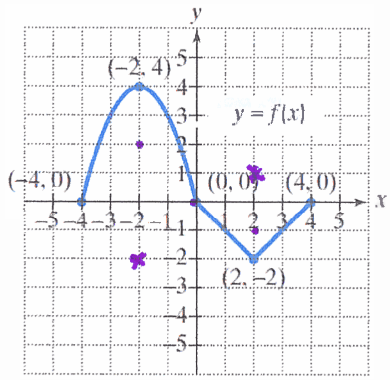
$g(x) = 3f(x)$

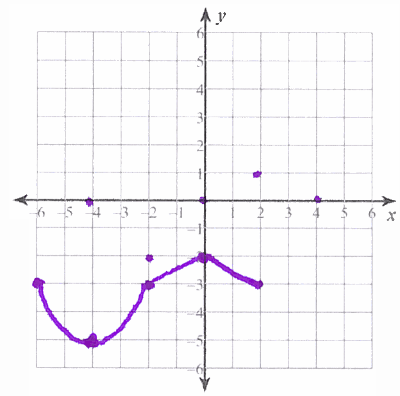
❖ ***ORDER IS IMPORTANT!***

1 ⇨	2 ⇨	3 ⇨	4 ⇨	5
REFLECTION ABOUT Y-AXIS	HORIZONTAL TRANSLATION	VERTICAL STRETCH/ compression	REFLECTION ABOUT X-AXIS	VERTICAL TRANSLATIONS

4. Let $y = f(x)$ be the function whose graph is given. Describe the transformations and then sketch the graphs of the transformations.

$$y = -\frac{1}{2}f(x+2) - 3$$





Transformations:

Reflect over x-axis
comp. x 1/2
Left 2
Down 3

5. The domain of a function $h(x)$ is $[0, 12]$ and its range is $[-4, 2]$.
What is the domain and range of $-2h(x+1) - 3$?

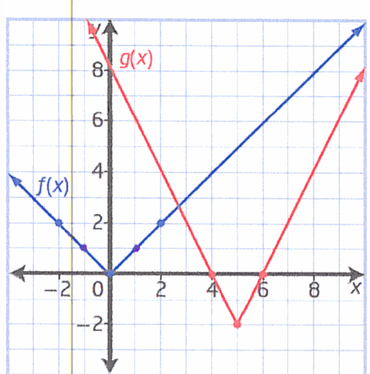
Reflect x-axis = −1·y
stretch x2 = 2y
Left 1 = x−1
Down 3 = y−3

D	R
x−1	−2y−3
[−1, 11]	[5, −7]

[−7, 5]

Write an equation for $g(x)$ as a transformation of the function $f(x)$.

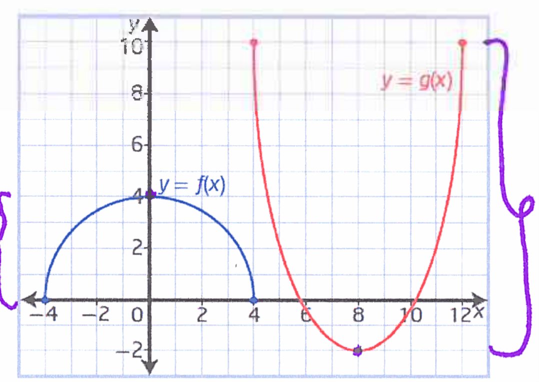
6.



stretch x 2
Right 5
Down 2
g(x) = 2f(x−5)−2

7.

h=4



h=12

Right 8
stretch x 3
Reflect over x-axis
up 10
g(x) = −3f(x−8)+10

· (8, −12)

7.1 – COMBINATIONS OF FUNCTIONS

OBJECTIVES: Combine functions using the algebra of functions
Evaluate the combination of functions for a given value

❖ THE ALGEBRA OF FUNCTIONS

➢ Let f and g be two functions...

Operation	Definition	Example Let $f(x) = 2x$ and $g(x) = -x + 5$.
Addition	$(f+g)(x) = f(x) + g(x)$	$2x + (-x + 5) = x + 5$
Subtraction	$(f-g)(x) = f(x) - g(x)$	$2x - (-x + 5) = 3x - 5$
Multiplication	$(f \cdot g)(x) = f(x) \cdot g(x)$	$2x(-x + 5) = -2x^2 + 10x$
Division	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},\ g(x) \neq 0$	$\frac{2x}{-x+5},\ x \neq 5$

EXAMPLES:

Use the given table to evaluate each given function.

x	-2	0	2	4	6
$f(x)$	1	5	11	19	29
$g(x)$	5	1	5	17	37

1. $(f+g)(4)$

$19+17=36$

2. $(fg)(-2)$

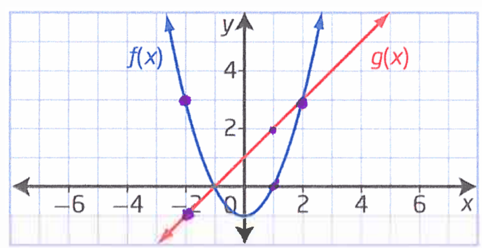
$1 \cdot 5 = 5$

3. $(g-f)(6)$

$37-29=8$

4. $\left(\frac{f}{g}\right)(2)$ $= \frac{11}{5}$

Use the given graphs to evaluate each given function.



5. $(f+g)(-2)$

$3+(-1)=2$

6. $(fg)(2)$

$3 \cdot 3 = 9$

7. $(g-f)(1)$

$2-0=2$

For the given functions $f(x) = 3x - 2$ & $g(x) = 2x^2$, evaluate...

8. $(f+g)(3)$

$f(3)=7$
$g(3)=18$
$(f+g)(3)=7+18=25$

9. $(f-g)(4)$

$f(4)=10$
$g(4)=32$
$(f-g)(4)=10-32=-22$

10. $(fg)(2)$

$f(2)=4$
$g(2)=8$
$(fg)(2)=4 \cdot 8=32$

Let $f(x) = x + 1$ & $g(x) = x^2 - 4$. Write a formula for the function.

11. $j(x) = g(x) - 2f(x)$

$x^2-4-2(x+1)$
$x^2-4-2x-2$
x^2-2x-6

12. $k(x) = f(x)g(x)$

$(x+1)(x^2-4)$
x^3+x^2-4x-4

13. $m(x) = [f(x)]^2 + g(x)$

$(x+1)^2+x^2-4$
$x^2+2x+1+x^2-4$
$2x^2+2x-3$

7.2 – INVERSE FUNCTIONS

OBJECTIVES: Evaluate the inverse of a function for a given value
Write the formula for an inverse function

❖ INVERSE FUNCTIONS

➢ If f is a one-to-one function with domain D and range R, then the inverse function of f, denoted f^{-1}, is the function with domain R and range D defined by: $a = f^{-1}(b)$ if and only if $b = f(a)$

Function: f(input) = output **Inverse function: f^{-1}(output) = input**

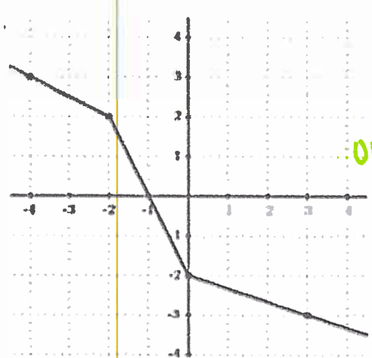
EXAMPLES: EVALUATING A FUNCTION & ITS INVERSE

Use the table of $g(t)$ to identify the missing function values.

1. $g(0) = ?$ 3
2. $g(?) = 0$ 4
3. $g(2) = ?$ 1
4. $g^{-1}(0) = ?$ 4 output
5. $g^{-1}(?) = 0$ 3 input
6. $g^{-1}(4) = ?$ −1 output

t	-3	-1	0	2	4	7
g(t)	6	4	3	1	0	-2

Use the function $f(x)$ graphed below to find the missing values.



7. $f(3) = ?$ -3
8. $f^{-1}(3) = ?$ -4 output
9. $f(0) = ?$ -2
10. $f^{-1}(0) = ?$ -1 output

Let $f(x) = 3x + 1$. Find the following:

11. $f(-1)$ $= 3(-1)+1 = -2$
12. $f^{-1}(x) = 13$ input $3(13)+1 = 40$
13. $f^{-1}(13)$ output $3x+1=13$, $3x=12$, $x=4$

❖ HOW TO FIND THE INVERSE OF A FUNCTION ALGEBRAICALLY

➢ Given a formula for a function $f(x)$, proceed as follows to find a formula for $f^{-1}(x)$

- Replace $f(x)$ with y
- Swap the x and the y
- Solve the function for y

EXAMPLES: FINDING THE INVERSE OF A FUNCTION

For the given function, find a formula for its inverse function.

14. $f(x) = \frac{1}{5}x + 2$ y

$x = \frac{1}{5}y + 2$
$-2 \quad -2$
$5(x-2) = \left(\frac{1}{5}y\right)5$
$5x - 10 = y$
$f^{-1}(x) = 5x - 10$

15. $f(x) = 4x^3 - 8$ y

$x = 4y^3 - 8$
$+8 \quad +8$
$\frac{x+8}{4} = \frac{4y^3}{4}$
$\sqrt[3]{\frac{x+8}{4}} = \sqrt[3]{y^3}$
$f^{-1}(x) = \sqrt[3]{\frac{x+8}{4}} = y$

16. $f(x) = \sqrt{5x+4}$ y

$x^2 = \left(\sqrt{5y+4}\right)^2$
$x^2 = 5y + 4$
$-4 \quad -4$
$\frac{x^2-4}{5} = \frac{5y}{5}$
$f^{-1}(x) = \frac{x^2-4}{5} = y$

7.3 – COMPOSITION OF FUNCTIONS

OBJECTIVES: Write a composition of two functions
Evaluate a composition of functions

❖ COMPOSITION OF FUNCTIONS

- Two functions connected by the fact that the output of one is the input of the other.
- For two functions $f(x)$ and $g(x)$, the function $f(g(x))$ is said to be a composition of f with g.
  - The function $f(g(x))$ is defined by using the output of the function g as the input to the function f.

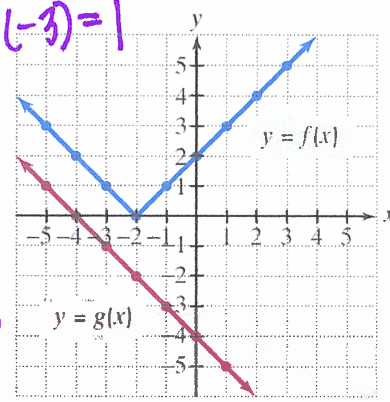
EXAMPLES: EVALUATING COMPOSITE FUNCTIONS

1. Given the functions $p(x) = 3 + \sqrt{x+5}$ and $q(x) = 2 + (x-1)^2$, find $q(p(-1))$.

$p(-1) = 3 + \sqrt{-1+5} = 3 + \sqrt{4} = 5$

$q(5) = 2 + (5-1)^2 = 2 + 4^2 = 18$

2. Use the graphs of f and g to evaluate each composite function.



$f(g(-1)) = f(-3) = 1$

$f(g(1))$
$= f(-5) = 3$

$g(f(0))$
$= g(2) = -6$

$g(f(-1))$
$= g(1) = -5$

3. The functions j and k are defined by the following sets of input and output values:

$$j = \{(0,-2), (4,1), (3,5), (5,0)\}$$
$$k = \{(1,2), (-2,4), (5,5), (6,-2)\}$$

4. Find: $k(j(4))$ $\quad$ $j(k(5))$

$= k(1) = 2$ $\quad$ $= j(5) = 0$

EXAMPLES: WRITING COMPOSITE FUNCTIONS

4. Let $f(x) = 4x^2 - 2$ and $g(x) = -3x + 1$.

Find a formula for $g(f(x))$.

$g(4x^2-2) = -3(4x^2-2) + 1$
$= -12x^2 + 6 + 1$
$= -12x^2 + 7$

Find a formula for $f(g(x))$.

$f(-3x+1) = 4(-3x+1)^2 - 2$
$= 4(9x^2 - 6x + 1) - 2$
$= 36x^2 - 24x + 4 - 2$
$= 36x^2 - 24x + 2$

5. Let $f(x) = \dfrac{2}{x^2 - 1}$ & $g(x) = \sqrt{1-5x}$ Find a formula for $f(g(x))$.

$f(\sqrt{1-5x}) = \dfrac{2}{(\sqrt{1-5x})^2 - 1} = \dfrac{2}{1-5x-1} = \dfrac{2}{-5x}$