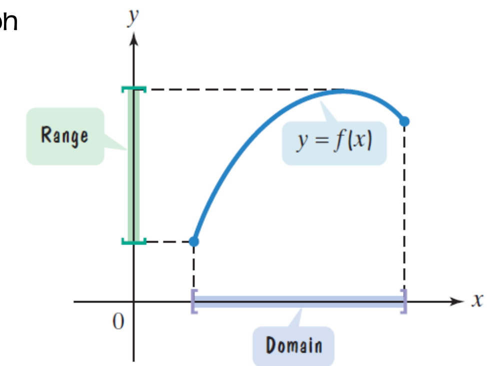
Name: ________________________________

Chapter 3: Functions & Their Transformations

3.APK.1 – DOMAIN & RANGE

OBJECTIVE: Determine the domain and range of a function given its graph



❖ Functions, Domain, & Range

- A function from a set D to a set R is a rule that assigns to element in D a unique element of R
  - Domain—the set D of all inputs; the variable x
  - Range—the set R of all outputs, the variable y

❖ Finding the Domain & Range of a Function

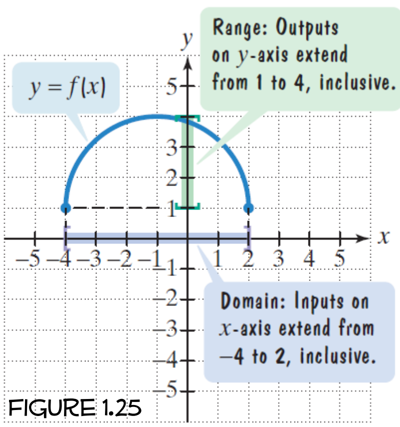


FIGURE 1.25

DOMAIN:

Using Set-Builder Notation

$\{ x \mid -4 \le x \le 2 \}$

The set of all x | such that | x is greater than or equal to −4 and less than or equal to 2.

Using Interval Notation

$[-4, 2]$.

The square brackets indicate −4 and 2 are included. Note the square brackets on the x-axis in **Figure 1.25**.

RANGE:

$\{ y \mid 1 \le y \le 4 \}$

The set of all y | such that | y is greater than or equal to 1 and less than or equal to 4.

$[1, 4]$.

The square brackets indicate 1 and 4 are included. Note the square brackets on the y-axis in **Figure 1.25**.

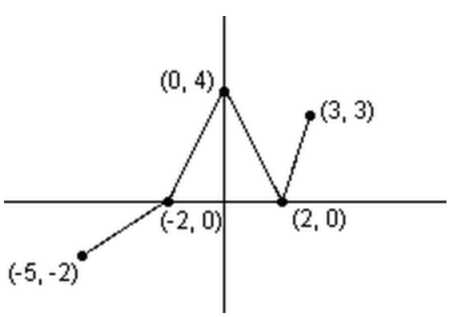
❖ Representing w/ Interval Notation

$(-\infty, \_)$ $(\_, \infty)$ [Brackets] (Parentheses)

EXAMPLES: FINDING THE DOMAIN & RANGE OF A FUNCTION GRAPHICALLY

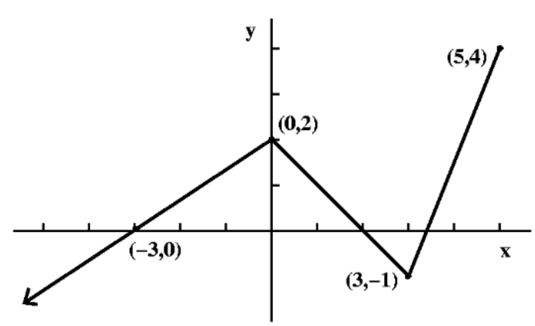
Use the graph to determine the domain and range of the function.

1.



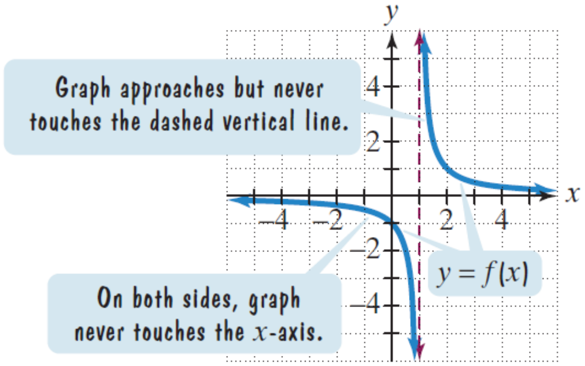
DOMAIN	RANGE

2.



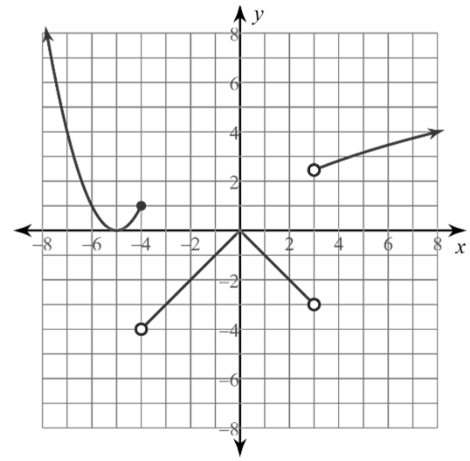
DOMAIN	RANGE

3.



DOMAIN	RANGE

4.



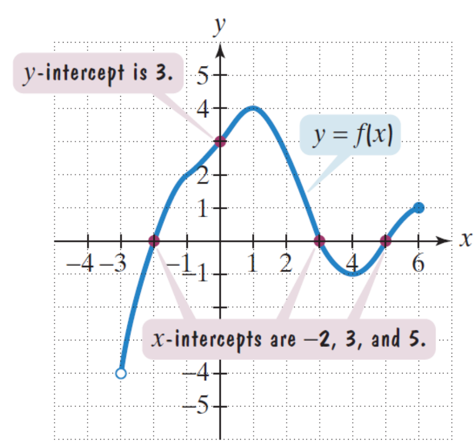
DOMAIN	RANGE

3.APK.2 – CHARACTERISTICS OF FUNCTIONS

OBJECTIVE: Analyze the graph of a function for: domain, range, x-intercept(s), y-intercept; intervals on which a function is increasing, decreasing, or constant; maximum & minimum values; and end behavior

❖ CHARACTERISTICS OF FUNCTIONS

- **x-intercept(s)** – where the graph crosses/touches the x-axis
  - Value at $y = 0$ or $f(x) = 0$
- **y-intercept** – where the graph crosses the y-axis
  - Value of the function at $x = 0$
- **Increasing/Decreasing/Constant Intervals**
  - Use ONLY the x-values of the function to describe the interval
    - x-values represent location while y-values represent values of the function
    - (x start, x end)
  - Use parentheses; never use brackets.
    - If we use brackets – and not parentheses – then we are saying that the value of the function is both increasing and decreasing at the same time at the same location.
- **Local Extrema: Maxima & Minima**
  - Occur ONLY at points at which a function changes its increasing or decreasing behavior.

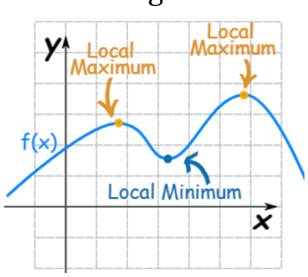


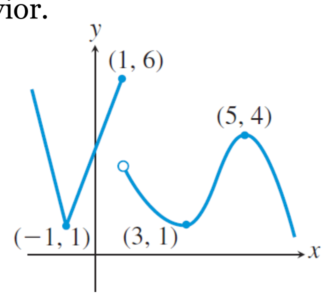
INCREASING INTERVALS:

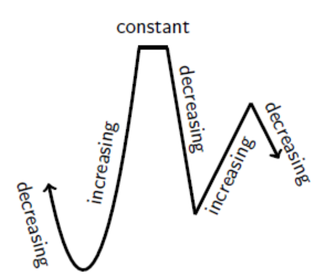
DECREASING INTERVAL:

MAX OF _______ AT X = _______

MIN OF _______ AT X = _______

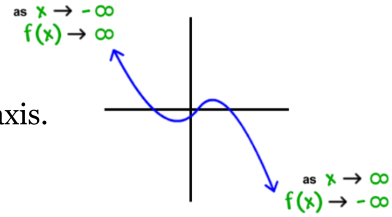


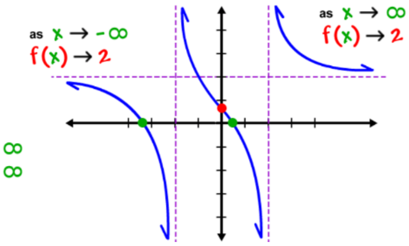




- **End behavior**

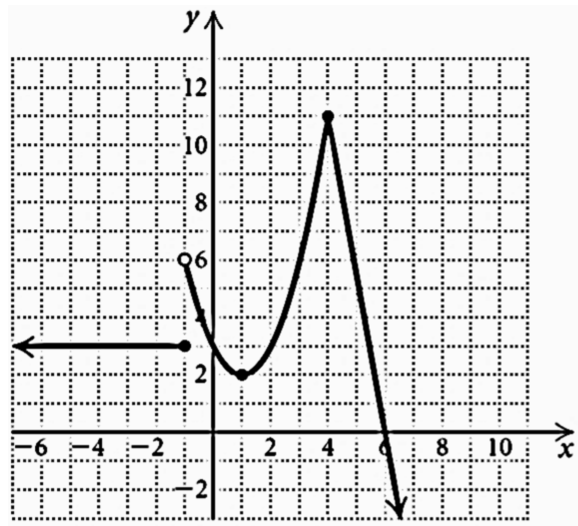
  The end behavior of a function describes the behavior of the graph at the "ends" of the x-axis.
  - Left: $\lim_{x\to-\infty} f(x) =$ a y value
  - Right: $\lim_{x\to\infty} f(x) =$ a y value





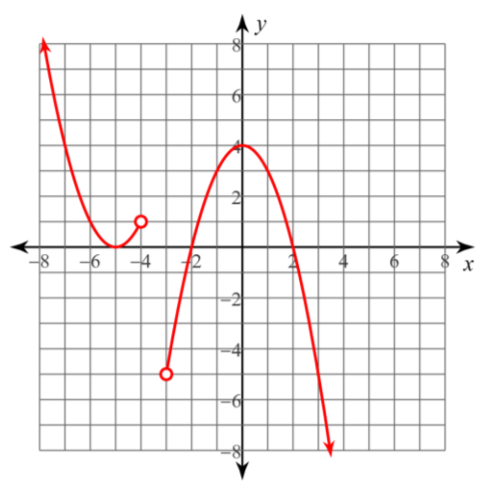
<u>EXAMPLES: ANALYZING FUNCTIONS</u>

Use the graph of $f(x)$ to find the following.



1. The domain:
2. The range:
3. The x-intercept:
4. The y-intercept:
5. Increasing interval(s):
6. Decreasing interval(s):
7. Constant interval(s):
8. Maximum value = _____; location: $x =$ _____
9. Minimum value = _____; location: $x =$ _____
10. $f(-10)$
11. Value(s) for which $f(x) = 6$
12. End behavior: $\lim_{x\to-\infty} f(x) =$
13. End behavior: $\lim_{x\to\infty} f(x) =$

Use the graph of $f(x)$ to find the following.



14. The domain
15. The range
16. The x-intercept(s)
17. The y-intercept
18. Interval(s) on which f is increasing
19. Interval(s) on which f is decreasing
20. End behavior: $\lim_{x\to-\infty} f(x) =$
21. End behavior: $\lim_{x\to\infty} f(x) =$

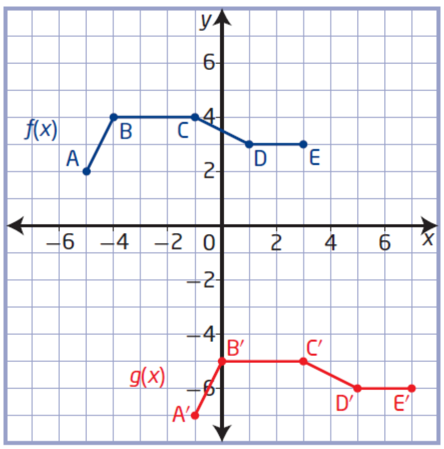
3.1 – VERTICAL & HORIZONTALS SHIFTS

OBJECTIVES:
- Identify the effect on the graph of a function replacing $f(x)$ by $f(x)+k$ and $f(x+k)$ for specific values of k (both positive or negative)
- Describe, write a formula, graph and interpret a function that has been shifted vertically and/or horizontally

❖ EXPLORING TRANSLATIONS – *What do you notice? What do you wonder?*

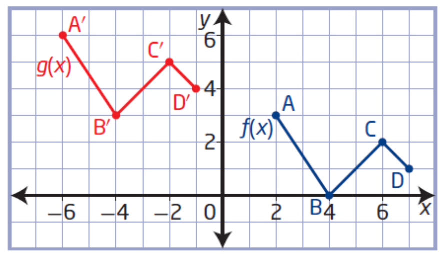
Also identify the domain and range of $f(x)$, as well as the domain and range of the transformed graphs.

$g(x) = f(x-4) - 9$



	DOMAIN	RANGE
$f(x)$		
$g(x)$		

$g(x) = f(x+8) + 3$



	DOMAIN	RANGE
$f(x)$		
$g(x)$		

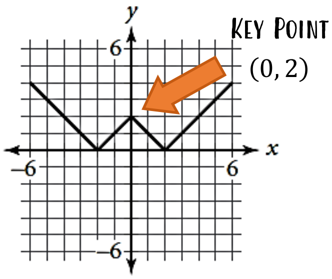
❖ TRANSLATIONS (shifts)

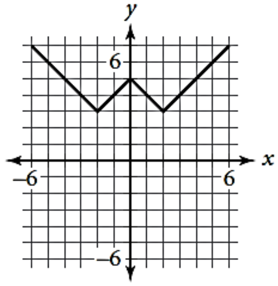
FUNCTION NOTATION	DESCRIPTION	COORDINATE RULE	DOMAIN OR RANGE CHANGE?
$y = f(x-h)$			
$y = f(x+h)$			
$y = f(x) + k$			
$y = f(x) - k$			

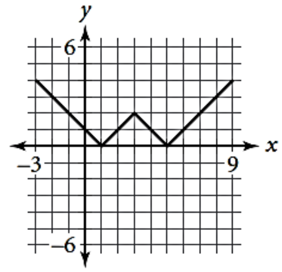
Examples:

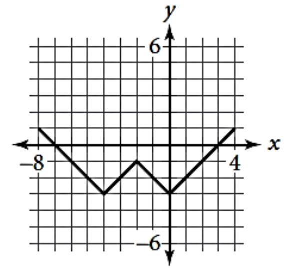
1. The graph of $g(x)$ contains the point $(-3, 0)$. Describe the translation and then write a formula for a translation of g that has a graph containing the point $(5, 9)$.

2. The graph of $y = f(x)$ is shown below. Write an equation for each related graph showing how the function has been translated.









3. Suppose that the x-intercepts of the graph of $f(x)$ are -5 & 3. What are the x-intercepts of the graph of $y = f(x + 2)$?

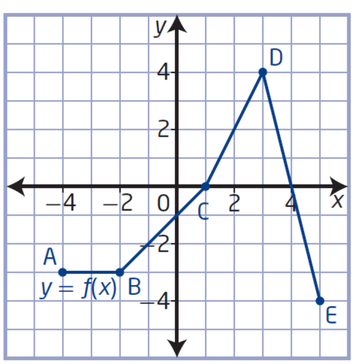
4. The domain of a function $h(x)$ is $[0, 12]$ and its range is $[-4, 2]$. What is the domain and range of $h(x + 5) - 12$?

3.2 – VERTICAL & HORIZONTALS REFLECTIONS

OBJECTIVES:

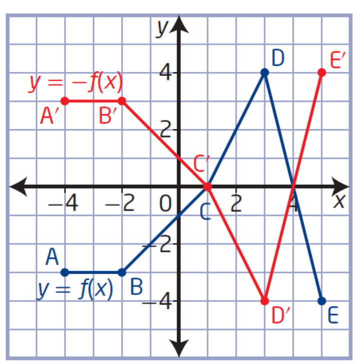
- Identify the effect on the graph of a function replacing $f(x)$ by $-f(x)$ and $f(-x)$
- Describe, write a formula, graph and interpret a function that has been reflected vertically and/or horizontally

❖ EXPLORING REFLECTIONS – *What do you notice? What do you wonder?*



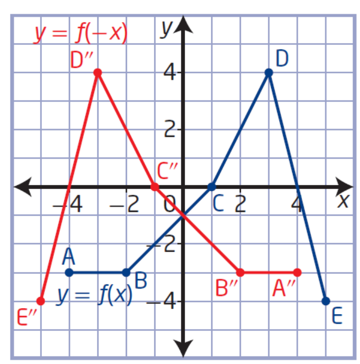
Graph $\boldsymbol{y = -f(x)}$

REFLECT OVER THE X-AXIS



Graph $\boldsymbol{y = f(-x)}$

REFLECT OVER THE Y-AXIS



COORDINATES OF A   COORDINATES OF E   COORDINATES OF A'   COORDINATES OF E'   COORDINATES OF A'   COORDINATES OF E'

❖ REFLECTIONS ACROSS AXES (flips)

FUNCTION NOTATION	DESCRIPTION	COORDINATE RULE	DOMAIN OR RANGE CHANGE?
$\boldsymbol{y = -f(x)}$	Reflect over x-axis		
$\boldsymbol{y = f(-x)}$	Reflect over y-axis		

Examples:

1. The graph of $f(x)$ contains the point $(2, -3)$. What point must lie on the reflected graph if the graph is…
   a. reflected about the x-axis?
   b. reflected about the y-axis?

2. The domain of a function $h(x)$ is $[0, 12]$ and its range is $[-4, 2]$.
   What is the domain and range of $-h(x-4)+5$?

❖ ORDER IS IMPORTANT!

1 ⇨ REFLECTION ABOUT Y-AXIS

2 ⇨ HORIZONTAL TRANSLATION

3 ⇨ VERTICAL STRETCH/ compression

4 ⇨ REFLECTION ABOUT X-AXIS

5 VERTICAL TRANSLATIONS

3. The graph of the parent function $f(x)$ is given. Match the transformed function with its graph.

$y = f(-x)$

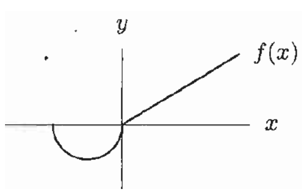
$y = -f(x)$

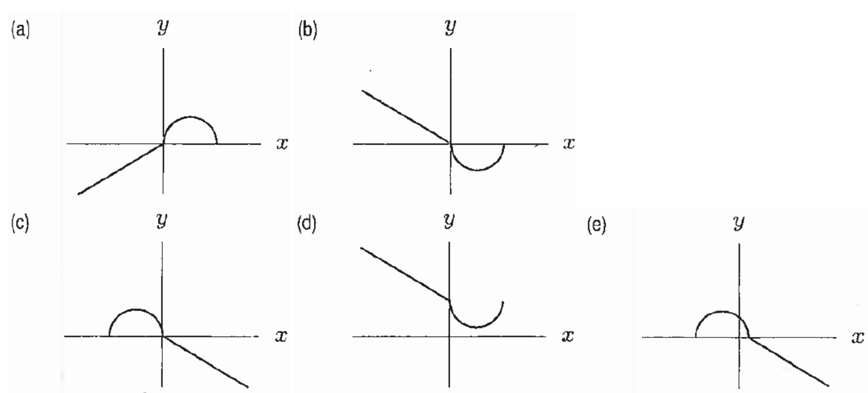
$y = f(-x) + 3$

$y = -f(x-1)$

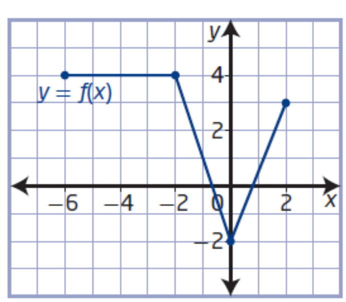
$y = -f(-x)$

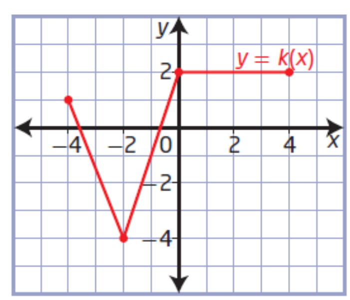
PARENT FUNCTION





4. The graph of $y = f(x)$ is shown at left. Describe the transformation and then write the equation of $k(x)$ in terms of $f(x)$.





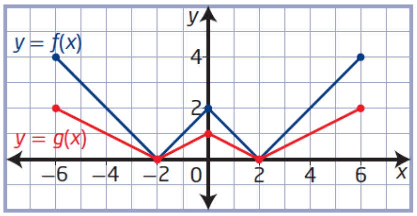
3.3 – Vertical Stretches & Compressions

OBJECTIVES:
- Identify the effect on the graph of a function replacing $f(x)$ by $kf(x)$ for specific values of k
- Describe, write a formula, graph and interpret a function that has been reflected vertically and/or horizontally

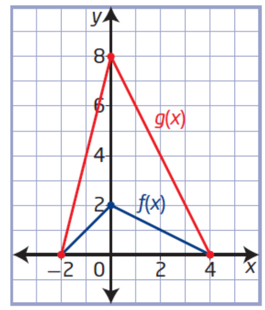
❖ Exploring Vertical Size Changes – *What do you notice? What do you wonder?*

Also identify the domain and range of $f(x)$, as well as the domain and range of the transformed graphs.

$g(x) = \frac{1}{2}f(x)$



$g(x) = 4f(x)$



$f(x)$		$g(x)$	
DOMAIN	RANGE	DOMAIN	RANGE

$f(x)$		$g(x)$	
DOMAIN	RANGE	DOMAIN	RANGE

❖ Vertical Stretches & Compressions

FUNCTION NOTATION	DESCRIPTION	COORDINATE RULE	DOMAIN OR RANGE CHANGE?
$y = Af(x), \lvert A\rvert > 1$			
$y = Af(x), 0 < \lvert A\rvert < 1$			

Examples:

1. The graph of $f(x)$ contains the point (3,–2). What corresponding point is on the graph of $g(x) = 3f(x-8)$?

2. The graph of $h(x)$ is found by vertically stretching the graph of $f(x)$ by a factor of 7, reflecting it about the x-axis, and then vertically shifting it down 3 units. Find a formula for $h(x)$ in terms of $f(x)$.

3. The function $g(x)$ is obtained from $f(x)$ by a single transformation. Use the tables below to find a formula for $g(x)$ in terms of $f(x)$.

x	-4	-2	0	2	4
$f(x)$	12	-4	-2	4	6

x	-4	-2	0	2	4
$g(x)$	36	−12	−6	12	18

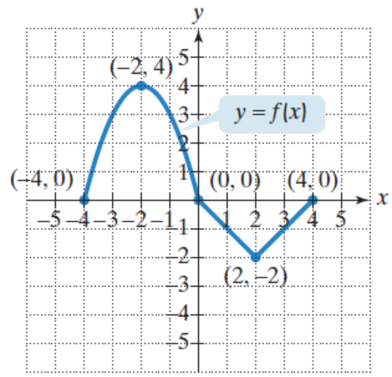
❖ ***ORDER IS IMPORTANT!***

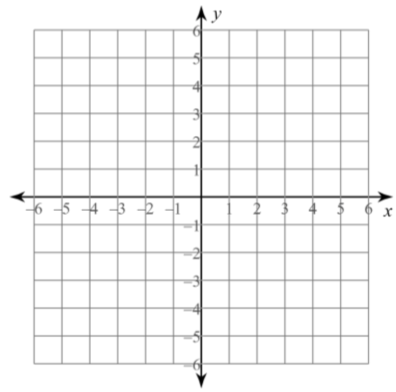
1 ⇨	2 ⇨	3 ⇨	4 ⇨	5
REFLECTION ABOUT Y-AXIS	HORIZONTAL TRANSLATION	VERTICAL STRETCH/ compression	REFLECTION ABOUT X-AXIS	VERTICAL TRANSLATIONS

4. Let $y = f(x)$ be the function whose graph is given. Describe the transformations and then sketch the graphs of the transformations.

$$y = -\frac{1}{2}f(x+2) - 3$$

Transformations:

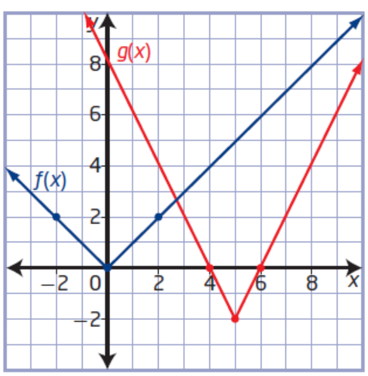




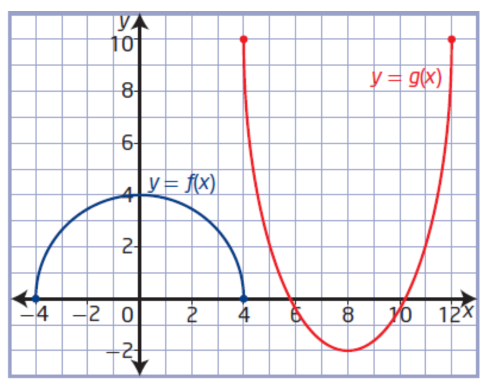
5. The domain of a function $h(x)$ is $[0, 12]$ and its range is $[-4, 2]$.
What is the domain and range of $-2h(x+1) - 3$?

Write an equation for $g(x)$ as a transformation of the function $f(x)$.

6.



7.



7.1 – Combinations of Functions

Objectives: Combine functions using the algebra of functions
Evaluate the combination of functions for a given value

❖ The Algebra of Functions

➢ Let f and g be two functions…

Operation	Definition	Example Let $f(x) = 2x$ and $g(x) = -x + 5$.
Addition	$(f + g)(x) = f(x) + g(x)$	$2x + (-x + 5) = x + 5$
Subtraction	$(f - g)(x) = f(x) - g(x)$	$2x - (-x + 5) = 3x - 5$
Multiplication	$(f \cdot g)(x) = f(x) \cdot g(x)$	$2x(-x + 5) = -2x^2 + 10x$
Division	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$	$\frac{2x}{-x+5}, x \neq 5$

Examples:

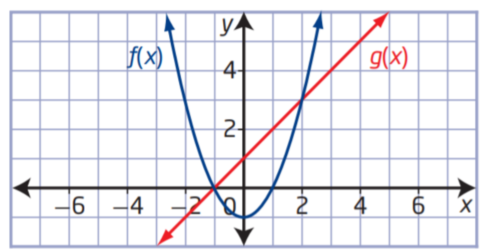
Use the given table to evaluate each given function.

1. $(f + g)(4)$
2. $(fg)(-2)$
3. $(g - f)(6)$
4. $\left(\frac{f}{g}\right)(2)$

x	-2	0	2	4	6
$f(x)$	1	5	11	19	29
$g(x)$	5	1	5	17	37

Use the given graphs to evaluate each given function.

5. $(f + g)(-2)$
6. $(fg)(2)$
7. $(g - f)(1)$



For the given functions $f(x) = 3x - 2$ & $g(x) = 2x^2$, evaluate…

8. $(f + g)(3)$
9. $(f - g)(4)$
10. $(fg)(2)$

Let $f(x) = x + 1$ & $g(x) = x^2 - 4$. Write a formula for the function.

11. $j(x) = g(x) - 2f(x)$
12. $k(x) = f(x)g(x)$
13. $m(x) = [f(x)]^2 + g(x)$

7.2 – Inverse Functions

Objectives: Evaluate the inverse of a function for a given value
Write the formula for an inverse function

❖ Inverse Functions

➢ If f is a one-to-one function with domain D and range R, then the inverse function of f, denoted f^{-1}, is the function with domain R and range D defined by: $a = f^{-1}(b)$ if and only if $b = f(a)$

Function: f(input) = output **Inverse function: f^{-1}(output) = input**

Examples: Evaluating a Function & its Inverse

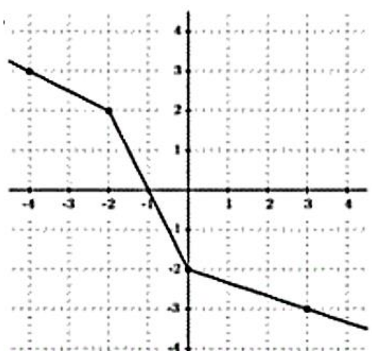
Use the table of $g(t)$ to identify the missing function values.

1. $g(0) = ?$
2. $g(?) = 0$
3. $g(2) = ?$

t	-3	-1	0	2	4	7
g(t)	6	4	3	1	0	-2

4. $g^{-1}(0) = ?$
5. $g^{-1}(?) = 0$
6. $g^{-1}(4) = ?$

Use the function $f(x)$ graphed below to find the missing values.



7. $f(3) = ?$
8. $f^{-1}(3) = ?$
9. $f(0) = ?$
10. $f^{-1}(0) = ?$

Let $f(x) = 3x + 1$. Find the following:

11. $f(-1)$
12. $f^{-1}(x) = 13$
13. $f^{-1}(13)$

❖ How to Find the Inverse of a Function Algebraically

➢ Given a formula for a function $f(x)$, proceed as follows to find a formula for $f^{-1}(x)$

- Replace $f(x)$ with y
- Swap the x and the y
- Solve the function for y

Examples: Finding the Inverse of a Function

For the given function, find a formula for its inverse function.

14. $f(x) = \frac{1}{5}x + 2$
15. $f(x) = 4x^3 - 8$
16. $f(x) = \sqrt{5x + 4}$

7.3 – Composition of Functions

Objectives: Write a composition of two functions
Evaluate a composition of functions

❖ Composition of Functions

- Two functions connected by the fact that the output of one is the input of the other.
- For two functions $f(x)$ and $g(x)$, the function $f\big(g(x)\big)$ is said to be a composition of f with g.
  - The function $f\big(g(x)\big)$ is defined by using the output of the function g as the input to the function f.

Examples: Evaluating Composite Functions

1. Given the functions $p(x) = 3 + \sqrt{x+5}$ and $q(x) = 2 + (x-1)^2$, find $q(p(-1))$.

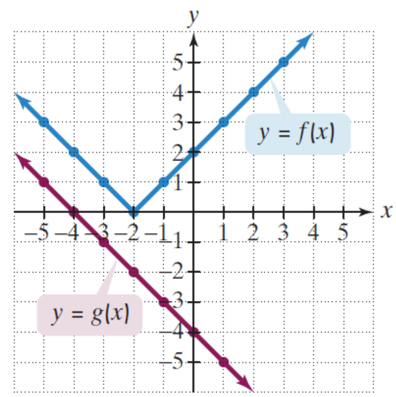
2. Use the graphs of f and g to evaluate each composite function.

$f(g(-1))$

$f(g(1))$

$g(f(0))$

$g(f(-1))$



3. The functions j and k are defined by the following sets of input and output values:

$$j = \{(0,-2), (4,1), (3,5), (5,0)\}$$
$$k = \{(1,2), (-2,4), (5,5), (6,-2)\}$$

4. Find: $k(j(4))$ $\quad$ $j(k(5))$

Examples: Writing Composite Functions

4. Let $f(x) = 4x^2 - 2$ and $g(x) = -3x + 1$.

   Find a formula for $g(f(x))$. $\quad$ Find a formula for $f(g(x))$.

5. Let $f(x) = \dfrac{2}{x^2 - 1}$ & $g(x) = \sqrt{1-5x}$ Find a formula for $f(g(x))$.