Chapter 5: Power, Polynomial & Rational Functions

PART 1 – POWER FUNCTIONS

5.2.D1 ~ Power Functions

OBJECTIVES:

- Interpret a function describing a power function given graphically, analytically, numerically, and verbally
- Identify power functions and write them in the form $f(x) = kx^a$

❖ POWER FUNCTIONS

- ➢ A function of the form $f(x) = kx^a$, where k and a are nonzero constants
  - ▪ k is the constant of variation & a is the power (exponent)
    - The exponent can take on any real-number value

Examples:

Determine whether the function is a power function. If it is, identify the power and the constant of variation.

1. $f(x) = \frac{1}{2}x^5$ 2. $f(x) = 3(2)^x$ 3. $f(x) = 6x^{-7}$

❖ PROPERTIES OF EXPONENTS

- ➢ Let $a, b, m, n,$ and x be real numbers:

$a^m a^n = a^{m+n}$ $(a^m)^n = a^{mn}$ $(ab)^m = a^m b^m$ $\sqrt{a} = a^{\frac{1}{2}}$ $\sqrt[n]{a^m} = a^{\frac{m}{n}}$

$\frac{a^m}{a^n} = a^{m-n}$ $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ $\frac{1}{a^m} = a^{-m}$ $\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$

- ▪ How these properties work:

The Product Rule

If b is a real number or algebraic expression, and m and n are integers,
$$b^m \cdot b^n = b^{m+n}.$$

- $2^2 \cdot 2^3 = 2^{2+3} = 2^5 = 32$
- $x^{-3} \cdot x^7 = x^{-3+7} = x^4$

When multiplying exponential expressions with the same base, add the exponents. Use this sum as the exponent of the common base.

The Power Rule

If b is a real number or algebraic expression, and m and n are integers,
$$(b^m)^n = b^{mn}.$$

- $(2^2)^3 = 2^{2 \cdot 3} = 2^6 = 64$
- $(x^{-3})^4 = x^{-3 \cdot 4} = x^{-12}$

When an exponential expression is raised to a power, multiply the exponents. Place the product of the exponents on the base and remove the parentheses.

Products Raised to Powers

If a and b are real numbers or algebraic expressions, and n is an integer,
$$(ab)^n = a^n b^n.$$

- $(-2y)^4 = (-2)^4 y^4 = 16y^4$
- $(-2xy)^3 = (-2)^3 x^3 y^3 = -8x^3 y^3$

When a product is raised to a power, raise each factor to that power.

The Quotient Rule

If b is a nonzero real number or algebraic expression, and m and n are integers,

$$\frac{b^m}{b^n} = b^{m-n}.$$

- $\frac{2^8}{2^4} = 2^{8-4} = 2^4 = 16$
- $\frac{x^3}{x^7} = x^{3-7} = x^{-4}$

When dividing exponential expressions with the same nonzero base, subtract the exponent in the denominator from the exponent in the numerator. Use this difference as the exponent of the common base.

Quotients Raised to Powers

If a and b are real numbers, $b \neq 0$, or algebraic expressions, and n is an integer,

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

- $\left(\frac{2}{5}\right)^4 = \frac{2^4}{5^4} = \frac{16}{625}$
- $\left(-\frac{3}{x}\right)^3 = \frac{(-3)^3}{x^3} = -\frac{27}{x^3} = -27x^{-3}$

When a quotient is raised to a power, raise the numerator to that power and divide by the denominator to that power.

Examples: Write the function in the form $f(x) = kx^a$ and identify the constant, k, and the power, a.

4. $f(x) = \frac{6}{x^5}$

5. $f(x) = (3x^{-2})^2$

6. $f(x) = \frac{4x^4}{5x^{-5}}$

7. $f(x) = \left(\frac{2}{x^4}\right)^3$

8. $f(x) = \sqrt{\frac{25}{x^5}}$

9. $f(x) = \frac{(x^3)^{-4}}{2x^3 \cdot x^2 \cdot x}$

5.2.D2 ~ Power Functions & Variation

OBJECTIVES:

- Interpret a function describing a power function given graphically, analytically, numerically, and verbally
- Write and interpret a function describing a direct, inverse, combined or joint variation relationship

- Variation
  - **DIRECT VARIATION**
    - $\boldsymbol{y = kx^a}$ with $a > 0$
    - "y varies directly with x^a"
    - "y is directly proportional to x^a"
  - **INVERSE VARIATION**
    - $\boldsymbol{y = \frac{k}{x^a}}$ with $c > 0$
    - "y varies inversely with x^c"
    - "y is inversely proportional to x^c"
  - **COMBINED VARIATION**
    - Direct variation & inverse variation occur at the same time
    - Example: $\boldsymbol{y = \frac{kx}{z}}$
    - y varies directly as x and inversely as z
  - **JOINT VARIATION**
    - A variation in which a variable varies directly as the product of two or more other variables
    - $\boldsymbol{y = kxz}$
    - "y varies jointly as x and z"

Examples:

Write a power function representing the verbal statement. Let k represent the constant of variation.

1. x varies jointly as y and the square of z
2. x varies directly as the cube of z and inversely as y
3. The distance, d, of an object from a planet is inversely proportional to the square root of the gravitational force, F, that the planet exerts on the object.
4. The work done, W, in stretching a spring is directly proportional to the square of the distance, d, that it is stretched.

- Applications Involving Variation
  - In many applications, the constant of proportionality, k, is unknown. The procedure for determining k is:
    - Write the equation relating the given variables, using k for the constant of proportionality.
    - Substitute the given values in for the variables & solve for k.
    - Rewrite the equation (from step 1) by substituting the value of k determined in step 3.

Examples:

5. The length, L (in feet), of skid distance left by a car varies directly as the square of the initial velocity, v (in miles per hour), of the car. A car traveling at 40 mph leaves skid distance of 40 feet. Determine the length of the skid distance left by a car traveling 60 mph.

INITIAL EQUATION	CONSTANT OF PROPORTIONALITY	NEW EQUATION	SOLUTION

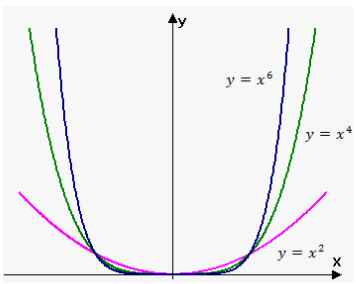
6. One's intelligence quotient (or IQ), I, varies directly as a person's mental age, m, and inversely as that person's chronological age, c. A person with a mental age of 25 and a chronological age of 20 has an IQ of 125. What is the chronological age of a person with a mental age of 40 & an IQ of 80?

INITIAL EQUATION	CONSTANT OF PROPORTIONALITY	NEW EQUATION	SOLUTION

❖ INVESTIGATING POWER FUNCTIONS, $f(x) = kx^a$, and the Effect the of Power a

➢ Positive Integer Powers, $a > 0$

Even Powers



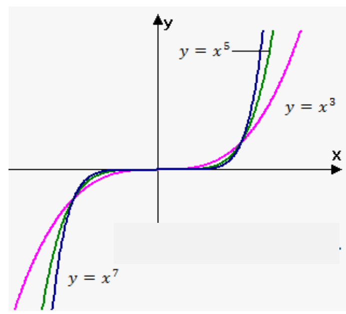
END BEHAVIOR:

$\lim_{x\to-\infty} x^E =$ ______

$\lim_{x\to\infty} x^E =$ ______

DOES THE GRAPH PASS THROUGH THE ORIGIN OR IS IT ASYMPTOTIC TO BOTH THE X-AXIS AND Y-AXIS?

Odd Powers



END BEHAVIOR:

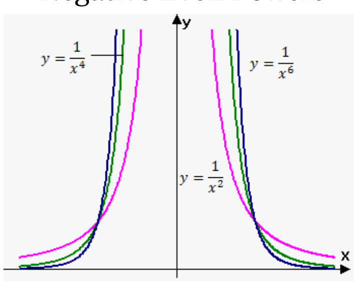
$\lim_{x\to-\infty} x^O =$ ______

$\lim_{x\to\infty} x^O =$ ______

DOES THE GRAPH PASS THROUGH THE ORIGIN OR IS IT ASYMPTOTIC TO BOTH THE X-AXIS AND Y-AXIS?

➢ Negative Integer Powers, $a < 0$

Negative Even Powers



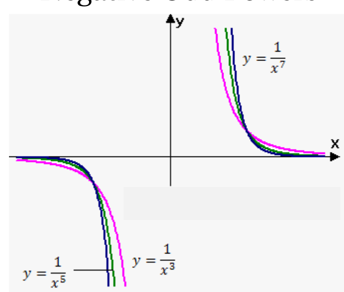
END BEHAVIOR:

$\lim_{x\to-\infty} x^{-E} =$ ______

$\lim_{x\to\infty} x^{-E} =$ ______

DOES THE GRAPH PASS THROUGH THE ORIGIN OR IS IT ASYMPTOTIC TO BOTH THE X-AXIS AND Y-AXIS?

Negative Odd Powers



END BEHAVIOR:

$\lim_{x\to-\infty} x^{-O} =$ ______

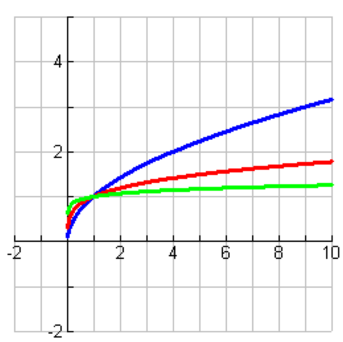
$\lim_{x\to\infty} x^{-O} =$ ______

DOES THE GRAPH PASS THROUGH THE ORIGIN OR IS IT ASYMPTOTIC TO BOTH THE X-AXIS AND Y-AXIS?

➢ Positive Rational Powers, $0 < a < 1$

- Rational exponents can be expressed using radical notation (and vice versa): $x^{\frac{1}{n}} = \sqrt[n]{x}$

Even Denominator Powers



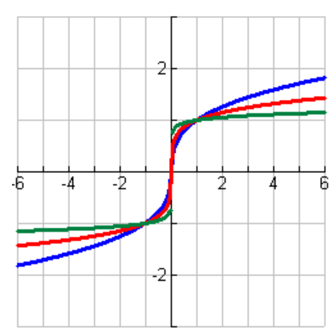
END BEHAVIOR:

$\lim_{x\to-\infty} x^{1/E} = \text{DNE}$

$\lim_{x\to\infty} x^{1/E} =$ ______

DOES THE GRAPH PASS THROUGH THE ORIGIN OR IS IT ASYMPTOTIC TO BOTH THE X-AXIS AND Y-AXIS?

Odd Denominator Powers



END BEHAVIOR:

$\lim_{x\to-\infty} x^{1/O} =$ ______

$\lim_{x\to\infty} x^{1/O} =$ ______

DOES THE GRAPH PASS THROUGH THE ORIGIN OR IS IT ASYMPTOTIC TO BOTH THE X-AXIS AND Y-AXIS?

5.2.D3 ~ Power Functions & Their Graphs

OBJECTIVES:

- Interpret a function describing a power function given graphically, analytically, numerically, and verbally
- Analyze the graphs of power function

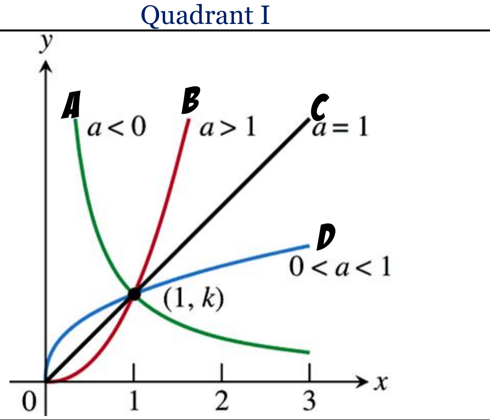
❖ POWER FUNCTIONS: END BEHAVIOR SUMMARY

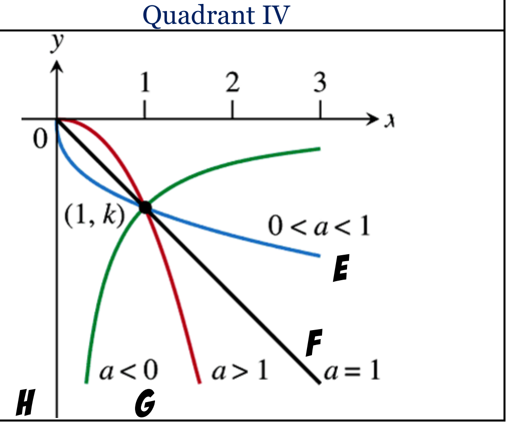
	x^E	x^O	x^{-E}	x^{-O}	$x^{1/E}$	$x^{1/O}$
$\lim_{x\to-\infty} f(x)$						
$\lim_{x\to\infty} f(x)$						

❖ GRAPHS OF POWER FUNCTIONS

- ➢ The graph contains the point $(1, k)$
  - ▪ If $a > 0$, then the graph passes through $(0, 0)$
  - ▪ If $a < 0$, then the graph is asymptotic to both axes
- ➢ There are FOUR possible shapes for general power functions of the form $f(x) = kx^a$, for $x \geq 0$

$k > 0$	$k < 0$
Quadrant I	Quadrant IV





Examples: If necessary, write the power function in the form $f(x) = kx^a$. Then, state the values of the constants k and a and describe the portion of the curve that lies in Quadrant I or Quadrant IV. Match the function to one of the curves (above). Describe the end behavior.

1. $f(x) = -2x^6$

$k =$ ______, $a =$ ______

GRAPH: ______ CONTAINS THE POINT: ______

PASSES THROUGH $(0, 0)$ ASYMPTOTIC TO BOTH AXES

$\lim_{x\to-\infty} f(x) =$ ______

$\lim_{x\to\infty} f(x) =$ ______

2. $f(x) = \dfrac{3}{4x^5}$

REWRITE IN THE FORM kx^a

$k =$ ______, $a =$ ______

GRAPH: ______ CONTAINS THE POINT: ______

PASSES THROUGH $(0, 0)$ ASYMPTOTIC TO BOTH AXES

$\lim_{x\to-\infty} f(x) =$ ______

$\lim_{x\to\infty} f(x) =$ ______

3. $f(x) = 3\sqrt[4]{x}$

REWRITE IN THE FORM kx^a $k =$ ______, $a =$ ______

GRAPH: ______ CONTAINS THE POINT: ______

PASSES THROUGH $(0,0)$ ASYMPTOTIC TO BOTH AXES

$\lim_{x\to-\infty} f(x) =$ ______

$\lim_{x\to\infty} f(x) =$ ______

4. $f(x) = \dfrac{-\sqrt[3]{x}}{2}$

REWRITE IN THE FORM kx^a $k =$ ______, $a =$ ______

GRAPH: ______ CONTAINS THE POINT: ______

PASSES THROUGH $(0,0)$ ASYMPTOTIC TO BOTH AXES

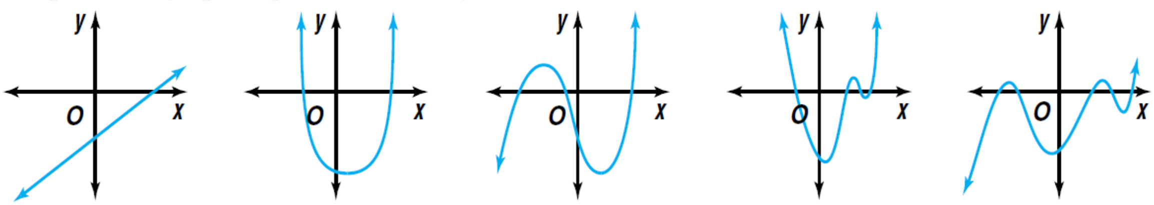
$\lim_{x\to-\infty} f(x) =$ ______

$\lim_{x\to\infty} f(x) =$ ______

PART 2 – POLYNOMIAL FUNCTIONS

INVESTIGATING POLYNOMIAL FUNCTIONS

- Graphs of Polynomial Functions
  - The shape of the graph depends on its degree, n.



Linear $n = 1$ | Quadratic $n = 2$ | Cubic $n = 3$ | Quartic $n = 4$ | Quintic $n = 5$

What do you notice about the degree and behavior – end behavior, number of x-intercepts, number of turning points – of the graph?

- Go to www.geogebra.org
  - Search for: **Unit 3 Investigation 3 Activity 2.1** *Author: Leigh Lessard*
- Use the graph to answer the following questions about the function: $P(x) = (x+2)^a(x-1)^b(x-3)^c$

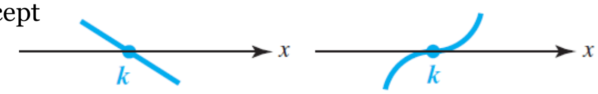
Intercept Form

The graph of $p(x) = a(x - x_1)(x - x_2)\ldots(x - x_n)$ has $x_1, x_2, \&\ x_n$ as it x-intercepts.

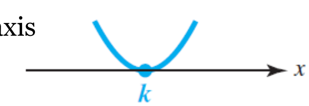
What are the x-intercepts of $P(x)$?

x-intercepts

The graph of $p(x)$ can cross the x-axis at an x-intercept



OR it can be *tangent* to the x-axis



Use the a slider to adjust the power of the linear factor $(x + 2)$.

For what powers of a does the graph cross the x-axis?

For what powers of a is the graph tangent to the x-axis?

5.1.D1 ~ Higher-Order Polynomial Functions

OBJECTIVES:

- Use the language of rate of change to describe the behavior of a higher-order polynomial function
- Sketch the graph of a polynomial function

❖ Cubic Functions

➢ Standard Form of a Cubic Function: $y = ax^3 + bx^2 + cx + d$

❖ **SUCCESSIVE DIFFERENCES** $= y_2 - y_1$

➢ Functions with constant first differences are linear; second differences are quadratic; & third differences are **cubic functions**.

❖ **CONCAVITY**

➢ The graph of a function f is said to be **CONCAVE UP** if its first differences *increase* as the input values increase.

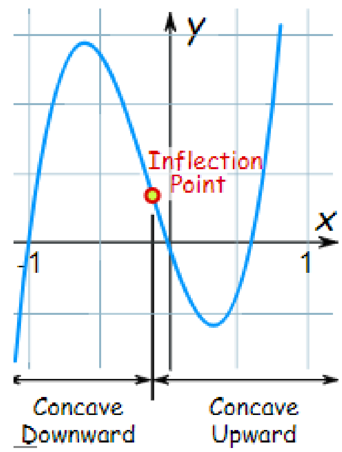
- Concave up functions curve upward.

➢ The graph of a function f is said to be **CONCAVE DOWN** if its first differences *decreases* as the input values increase.

- Concave down functions curve downward.

❖ **INFLECTION POINTS**

➢ The point on a graph where the function changes concavity.



Examples: Numerical representations of either a linear or quadratic function are shown in a table. Find successive rates of change to determine if the function is linear, quadratic, or cubic. Identify intervals where the function is increasing and/or decreasing and concave up, concave down, or neither.

1. $f(x)$

x	$f(x)$	FIRST DIFFERENCES	SECOND DIFFERENCES	THIRD DIFFERENCES
1	-1.00			
2	-0.92			
3	-0.68			
4	-0.28			
5	0.28			

LINEAR / QUADRATIC / CUBIC

INCREASING: ____ $\leq x \leq$ ____

DECREASING: ____ $\leq x \leq$ ____

CONCAVE UP

CONCAVE DOWN

NEITHER

2. $g(x)$

X	$g(x)$	FIRST DIFFERENCES	SECOND DIFFERENCES	THIRD DIFFERENCES
1	-0.88			
2	-3.04			
3	-5.76			
4	-8.32			
5	-10.00			

LINEAR / QUADRATIC / CUBIC

INCREASING: ____ $\leq x \leq$ ____

DECREASING: ____ $\leq x \leq$ ____

CONCAVE UP

CONCAVE DOWN

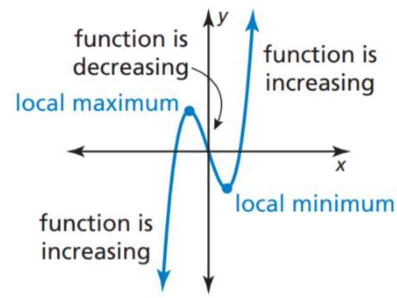
NEITHER

❖ POLYNOMIAL FUNCTIONS

- Linear, quadratic, and cubic functions are all polynomial functions!
  - The table summarizes the characteristics and appearance of graphs of polynomial functions:

FUNCTION NAME & DEGREE	CONCAVITY	INFLECTION POINTS	END BEHAVIOR	CONSTANT DIFFERENCE
Linear First degree	No concavity	No inflection points	One end $\to \infty$ One end $\to -\infty$	First
Quadratic Second degree	Concave up only or concave down only	No inflection points	Both ends $\to \infty$ or both ends $\to -\infty$	Second
Cubic Third degree	Concave up and concave down	One inflection point	One end $\to \infty$ One end $\to -\infty$	Third

❖ LOCAL EXTREMA OF POLYNOMIAL FUNCTIONS

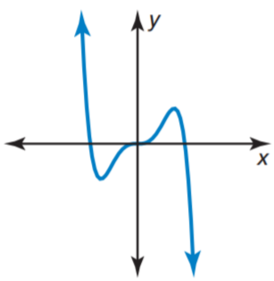


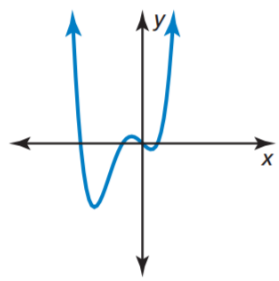
- The graph of a polynomial function of degree n will have ***AT MOST N – 1*** local extrema (turning points) but it may have fewer.

- A **local maximum** occurs at the point where the graph changes from increasing to decreasing.
- A **local minimum** occurs at the point where the graph changes from decreasing to increasing.

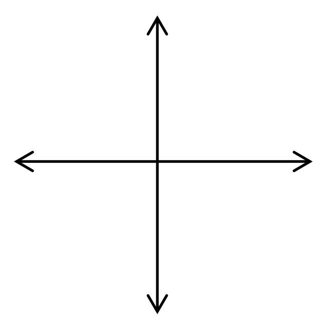
Examples:

3. Describe the degree and leading coefficient of the polynomial function using the graph:

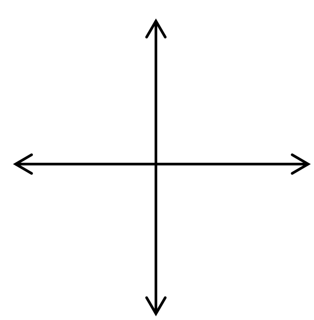




4. Graph a polynomial function that meets the following criteria:
   - x-intercepts of -4, 0, and 2
   - local maximum at $x = 1$
   - local minimum at $x = -2$



5. Graph a polynomial function that meets the following criteria:
   - Fourth degree
   - As $x \to \infty, f(x) \to \infty$
   - The graph of f has two inflection points
   - $f(x) = 0$ exactly twice
   - 1 minimum



5.1.D2 ~ The Behavior of Polynomial Functions

OBJECTIVES:

- Demonstrate an understanding of the characteristics, behaviors, formulas, graphs, and applications of polynomial functions
- Use the Leading Term Test to analyze the long-run/end behavior of polynomials
- Interpret a function describing a polynomial given graphically, analytically, numerically, and verbally
- Find x-intercepts/zeros, identify the multiplicity of zeros and describe its effect on the graph of a polynomial function

❖ POLYNOMIAL FUNCTIONS

- A **polynomial function** is a sum of power functions with nonnegative integer exponents.
- A **polynomial function of degree *n***, where n is a nonnegative integer, is defined by:

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

  - $a_n x^n$: **leading term**; n is the largest degree
  - a_n is the coefficient of the variable to the highest power – aka the **leading coefficient**
  - a_0 is the **constant coefficient term**, the initial value of the function, & the ***y*-intercept**

❖ Long-Run Behavior = End Behavior

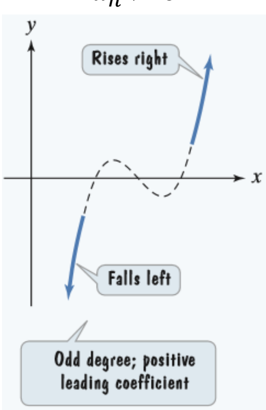
  - The behavior of the graph of a function to the far left ($x \to -\infty$) or the far right ($x \to \infty$) is called its **end behavior**.

➢ THE LEADING TERM TEST

  - End behavior depends upon the leading term: $a_n x^n$
  - The sign of the leading coefficient, a_n, and the degree, n, of the polynomial reveal its end behavior.

Degree n is odd

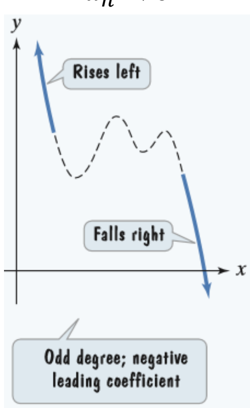
$a_n > 0$



$\lim_{x\to-\infty} f(x) = -\infty$

$\lim_{x\to\infty} f(x) = \infty$

$a_n < 0$

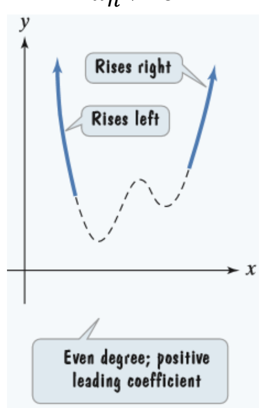


$\lim_{x\to-\infty} f(x) = \infty$

$\lim_{x\to\infty} f(x) = -\infty$

Degree n is even

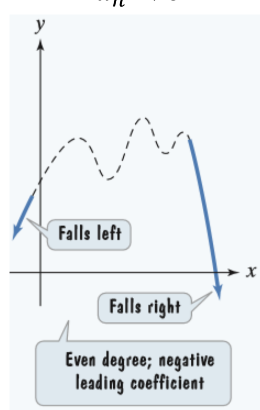
$a_n > 0$



$\lim_{x\to-\infty} f(x) = \infty$

$\lim_{x\to\infty} f(x) = \infty$

$a_n < 0$



$\lim_{x\to-\infty} f(x) = -\infty$

$\lim_{x\to\infty} f(x) = -\infty$

Examples: Use the Leading Term Test to determine the long-run/end behavior of the graph of the polynomial function.

	$\lim_{x\to-\infty} f(x)$	$\lim_{x\to\infty} f(x)$		$\lim_{x\to-\infty} f(x)$	$\lim_{x\to\infty} f(x)$
1. $f(x) = x^3 + 3x^2 - 3$			2. $f(x) = x^4 - 4x^2$		

FUNCTION	LEADING TERM	END BEHAVIOR $\lim_{x\to-\infty} f(x)$	END BEHAVIOR $\lim_{x\to\infty} f(x)$
What about this one? 3. $f(x) = -4x^3(x-1)^2(x+5)$			

Solution Although the equation for f is in factored form, it is not necessary to multiply to determine the degree of the function.

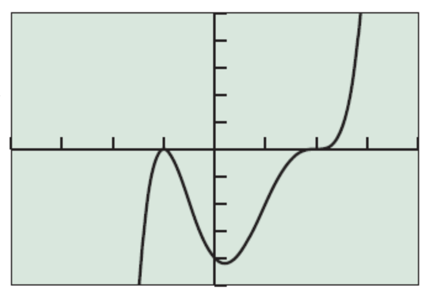
$$f(x) = -4x^3(x-1)^2(x+5)$$

Degree of this factor is 3. Degree of this factor is 2. Degree of this factor is 1.

When multiplying exponential expressions with the same base, we add the exponents. This means that the degree of f is $3 + 2 + 1$, or 6, which is even. Even-degree polynomial functions have graphs with the same behavior at each end. Without multiplying out, you can see that the leading coefficient is -4, which is negative. Thus, the graph of f falls to the left and falls to the right (↙, ↘).

FUNCTION	LEADING TERM	END BEHAVIOR $\lim_{x\to-\infty} f(x)$	END BEHAVIOR $\lim_{x\to\infty} f(x)$
4. $f(x) = -0.25x(x+6)^3(x-2)^4$			
5. $f(x) = -2x^3(x-1)(x+5)$			
6. $f(x) = -0.5(1-x)(2x-3)^2$			

- Introduction to Short-Run Behavior of Polynomials
  - The long-run behavior of a polynomial is determined by its leading term. **However, polynomials with the same leading term, may have different short-run behaviors.**
  - Intercept Form
    - The graph of $p(x) = a(x-x_1)(x-x_2)\ldots(x-x_n)$ has $x_1, x_2, \& x_n$ as it x-intercepts/zeros.
- ***MULTIPLICITY AND X-INTERCEPTS***
  - In factoring the equation for the polynomial function f, is the same factor $x - k$ occurs m times, we call k a **zero with multiplicity m**
  - If k is a zero of EVEN multiplicity:
    - The graph touches the x-axis and turns around at k
  - If k is a zero of ODD multiplicity:
    - The graph crosses the x-axis at k



$f(x) = (x+1)^2(x-2)^3$

Example: Find the zeros for the polynomial function and give the multiplicity for each zero. State whether the graph crosses the x-axis, or touches the x-axis and turns around, at each zero.

7. $f(x) = 0.5(x+1)(2x-3)^2$

ZERO	MULTIPLICITY	CROSSES	TOUCHES

8. $f(x) = -4(2x+1)^2(x-5)^3$

ZERO	MULTIPLICITY	CROSSES	TOUCHES

5.1.D3 ~ Graphing Polynomial Functions

OBJECTIVES:

- Demonstrate an understanding of the characteristics, behaviors, formulas, graphs, and applications of polynomial functions
- Interpret a function describing a polynomial given graphically, analytically, numerically, and verbally
- Find x-intercepts/zeros, identify the multiplicity of zeros and describe its effect on the graph of a polynomial function
- Sketch the graph of a polynomial function

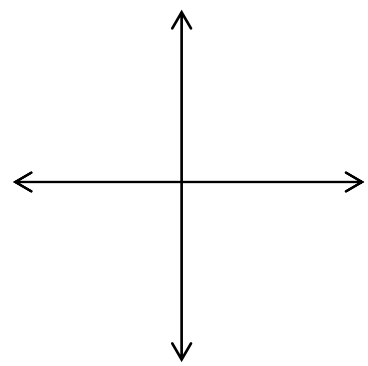
❖ GRAPHING A POLYNOMIAL FUNCTION

- ➢ Use the Leading Term Test to determine the graph's end behavior.
- ➢ Find x-intercepts (zeros) and determine whether the graph crosses or touches the x-axis.
- ➢ Find the y-intercept by computing $f(0)$.

Example: Analyze the polynomial function, $f(x)$, for its end behavior, x-intercepts/zeros, and y-intercepts and then sketch the graph by hand.

1. $f(x) = -4(x+2)(x-1)^2$

SKETCH:



END BEHAVIOR:

$\lim_{x \to -\infty} f(x) =$ $\lim_{x \to \infty} f(x) =$

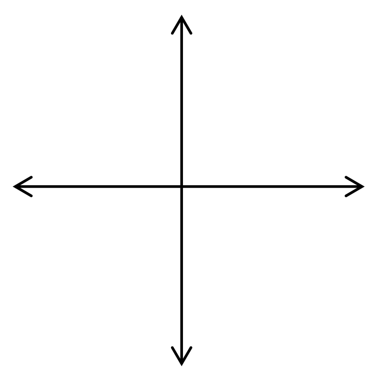
x-INTERCEPTS:

ZERO MULTIPLICITY CROSS OR TOUCH

y-INTERCEPT:

2. $f(x) = -0.5x(x+3)(x-4)^2$

SKETCH:



END BEHAVIOR:

$\lim_{x \to -\infty} f(x) =$ $\lim_{x \to \infty} f(x) =$

x-INTERCEPTS:

ZERO MULTIPLICITY CROSS OR TOUCH

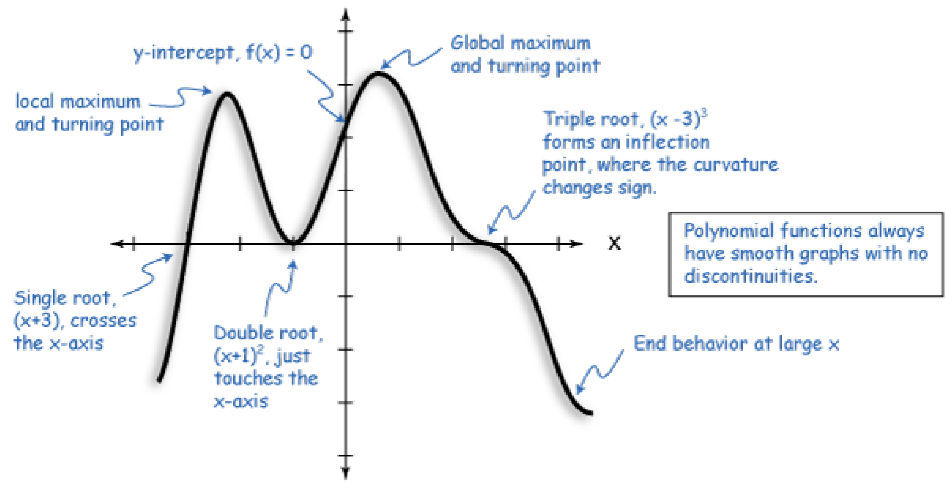
y-INTERCEPT:

5.1.D4 ~ Writing Polynomial Functions

OBJECTIVES:

- Demonstrate an understanding of the characteristics, behaviors, formulas, graphs, and applications of polynomial functions
- Write and interpret a function describing a polynomial given graphically, analytically, numerically, and verbally

The function shown below could be represented by the function $f(x) = -0.025(x+3)(x+1)^2(x-3)^3$, do you know why?

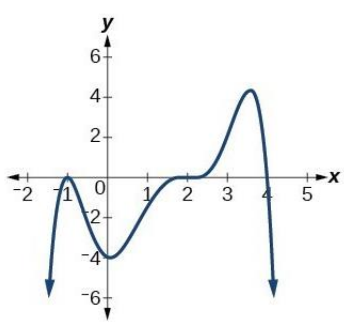


❖ FINDING THE FORMULA FOR A POLYNOMIAL FROM ITS GRAPH

- ➢ Determine the zeros/x-intercepts of the function and use multiplicity to determine whether the graph bounces off the x-axis (even multiplicity) or crosses it (odd multiplicity).
  - ▪ Write out the linear factorization: $y = a(x-k_1)(x-k_2)\cdots(x-k_n)$ including the multiplicity (if not 1).
- ➢ Plug the coordinates of a given point, or the y-intercept, into the linear factorization and solve to determine the value of a.

Examples: Find a possible formula for each polynomial whose graph is shown or described.

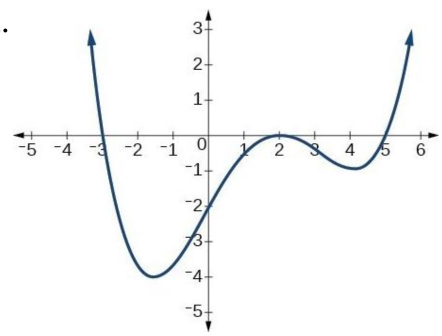
1.



X-INTERCEPTS	MULTIPLICITY	LINEAR FACTOR

LINEAR FACTORIZATION:

2.



X-INTERCEPTS	MULTIPLICITY	LINEAR FACTOR

LINEAR FACTORIZATION:

3. f is a fourth degree polynomial with a double zero at $x = 3, f(5) = 0, f(-1) = 0, \& f(0) = 3$

X-INTERCEPTS	MULTIPLICITY	LINEAR FACTOR

LINEAR FACTORIZATION:

5.1.D5 ~ Finding Zeros of Polynomial Functions

OBJECTIVES:

- Demonstrate an understanding of the characteristics, behaviors, formulas, graphs, and applications of polynomial functions
- Write and interpret a function describing a polynomial given graphically, analytically, numerically, and verbally
- Find x-intercepts/zeros of polynomial functions

- Short-Run Behavior of Polynomials
  - To determine the short-run behavior of a polynomial, we write it in intercept/factored form with as many linear factors as possible.
    - Intercept/factored form: $p(x) = a(x - x_1)(x - x_2) \dots (x - x_n)$
    - The values of x for which $p(x) = 0$ are called the **zeros**.
    - If the formula for p has a **linear factor**, a factor of the form $(x - k)$, then p has a zero at $x = k$.
      - Use factoring and the Zero Product Property to find the linear factors and zeros of polynomial functions.

- ***FACTORING BINOMIALS*** ALWAYS LOOK FOR COMMON FACTORS FIRST!
  - Difference of Two Squares: $a^2 - b^2 = (a + b)(a - b)$
  - Sum of Two Cubes: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
  - Difference of Two Cubes $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

$a = \sqrt[3]{a^3}$ & $b = \sqrt[3]{b^3}$

$$\mathbf{1^3 = 1, 2^3 = 8, 3^3 = 27, 4^3 = 64, 5^3 = 125}$$

Examples – Factor each binomial, completely.

1. $18x^2 - 2$
2. $81x^4 - 16$
3. $125x^3 + 64$
4. $32x^3 - 4$

❖ FACTORING POLYNOMIALS WITH 4 TERMS

- Factor by grouping
  - Group the first two terms & the last two terms
  - Factor out the common factor of each group
  - Write as a product of two binomials

Example – Factor to write the polynomial function in intercept form. Then use the Zero Product Property to find the zeros.

5. $f(x) = x^3 + 3x^2 - x - 3$

❖ FACTORING TRINOMIALS – SEE GOLD CARD

❖ QUADRATIC IN FORM: $x^4 + bx^2 + c$ OR $ax^4 + bx^2 + c$

- If you recognize how the expression factors in its original form, then do so.
- If $a \neq 1$, use the box method or the DEFOIL/A-C method.
  - Example:

$x^4 - 7x^2 - 18$

-18	
-9 & 2	-7

Find the factors of c that add up to be the middle term b

$(x^2 - 9)(x^2 + 2)$ — Write as a product of two binomials

$(x + 3)(x - 3)(x^2 + 2)$ — *Be on the lookout for a difference of two squares; the expression can be factored further.*

Examples – Factor to write the polynomial function in intercept form. Then use the Zero Product Property to find the zeros.

6. $f(x) = 4x^3 - 5x^2 - 9x$

7. $f(x) = -x^4 + 13x^2 - 36$

5.1.D6 ~ More Zeros of Polynomial Functions

OBJECTIVES:

- Demonstrate an understanding of the characteristics, behaviors, formulas, graphs, and applications of polynomial functions
- Write and interpret a function describing a polynomial given graphically, analytically, numerically, and verbally
- Divide polynomials using synthetic division

❖ Intercept Form, Zeros, & the Short-Run Behavior of a Polynomial

- ➢ ***LONG-RUN/END BEHAVIOR*** – predicted from the **degree**/highest-power term
  - Use the Leading Term Test
- ➢ ***SHORT-RUN BEHAVIOR*** – predicted from the zeros given by a linear factor
  - Use the Zero Product Property
  - Use the Quadratic Formula for quadratic factors that cannot be factored.

$$ax^2 + bx + c = 0 \rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- ➢ ***CONSTANT = Y-INTERCEPT*** – evaluate $f(0)$

Examples:

1. For the polynomial below...What is the degree? What is the leading coefficient? What is the constant coefficient? What is the long-run/end behavior? What are the roots (aka zeros) of the polynomial? *If necessary, round to 2 decimal places.*

$$f(x) = (5 - 2x)(x^2 - 10)(2 + 4x - x^2)$$

DEGREE/LEAD CO./CONSTANT

LONG-RUN/END BEHAVIOR

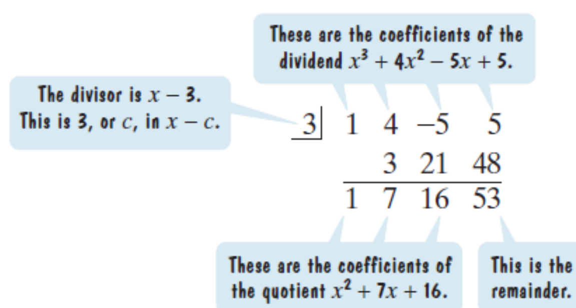
DEGREE	LEAD CO.	CONSTANT

$\lim_{x \to -\infty} f(x) =$ $\lim_{x \to \infty} f(x) =$

ROOTS/ZEROS:

❖ Synthetic Division

- ➢ Used to divide polynomials if the divisor is of the form $x - c$.
  - Divide $x^3 + 4x^2 - 5x + 5$ by $x - 3 \rightarrow$
    - Notice the relationship between the polynomials in the long division process and the numbers that appear in synthetic division.



❖ Synthetic Division

- ➢ To divide a polynomial by $x - c$:
  - Arrange in descending powers, with a 0 coefficient for any missing term.
  - Write c for the divisor, $x - c$.
  - Bring the first term down.
  - Multiply by c; add to the next term in the next column. *Repeat until the end.*
  - Use the numbers in the last row to write the quotient (add the remainder above the divisor).
    - *The degree of the first term of the quotient is one less than the degree of the first term of the dividend.*

Examples – Divide using synthetic division.

2. $(x^3 - 2x^2 - 5x + 6) \div (x - 3)$

3. $(5x^3 + 6x + 8) \div (x + 2)$

4. Solve the equation $15x^3 + 14x^2 - 3x - 2 = 0$ given that $-\mathbf{1}$ is a zero of the equivalent function.
   a. Synthetic division:

   b. Complete factorization:

   c. Zeros:

PART 3 – RATIONAL FUNCTIONS

INVESTIGATING RATIONAL FUNCTIONS

- Rational Functions
  - Rational functions are ratios (aka quotients) of polynomial functions: $r(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ & $q(x)$ are polynomial functions with $q(x) \neq 0$.

- Investigation I: Long-Run/End Behavior
  - ***Rational functions whose numerators and denominators have equal degrees:***

$$f(x) = \frac{5x+4}{x+2}$$

-100	-50	-25	-2.1	-2	-1.9	-1.5	10	50	100
5.06	5.13	5.26	65	Und.	-55	-7	4.5	4.88	4.94

$$g(x) = \frac{x+6}{2x+3}$$

-100	-50	-25	-1.8	-1.5	-1.3	-1	10	50	100
0.48	0.45	0.40	-7	Und.	11.75	5	0.70	0.54	0.52

End Behavior $\lim_{x \to -\infty} f(x) =$ ____ & $\lim_{x \to \infty} f(x) =$ ____ $\lim_{x \to -\infty} g(x) =$ ____ & $\lim_{x \to \infty} g(x) =$ ____

- What is the connection between the ratio of the leading terms of the numerator and denominator and the end behavior?

  - ***Rational functions where the degree of numerator < degree of the denominator:***

$$f(x) = \frac{3x+1}{x^2+x-2}$$

-300	-200	-100	-50	-2.1	1.1	50	100	200	300
-0.01	-0.02	-0.03	-0.06	-17.1	13.87	0.06	0.03	0.02	0.01

End Behavior $\lim_{x \to -\infty} f(x) =$ ____ & $\lim_{x \to \infty} f(x) =$ ____

- When the degree of the numerator is less than the degree of the denominator, the $\lim_{x \to \pm\infty} f(x) =$ ______.

  - ***Rational functions where the degree of numerator > degree of the denominator:***

$$g(x) = \frac{x^2+1}{2+x}$$

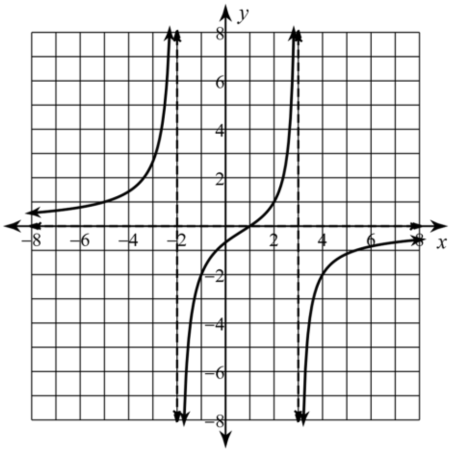
-300	-200	-100	-50	-2	1.1	50	100	200	300
-302	-202	-102	-52.1	Und.	0.71	48.1	98.05	198.02	298.02

End Behavior $\lim_{x \to -\infty} f(x) =$ ____ & $\lim_{x \to \infty} f(x) =$ ____

❖ Investigation 2: Long-Run & Short-Run Behavior

In this investigation, each rational function is represented in standard form, factored form, and graphically. Use the function formula & its graph to identify the long-run and short-run behavior of each rational function.

$$f(x) = \frac{-4x+4}{x^2-x-6} = \frac{-4(x-1)}{(x-3)(x+2)}$$

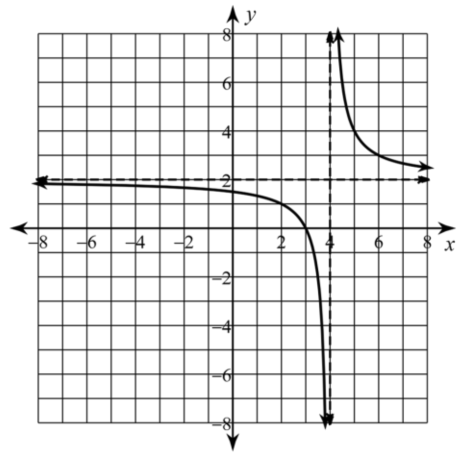


End behavior: $\lim_{x\to-\infty} f(x) =$ $\lim_{x\to\infty} f(x) =$

Horizontal asymptote: Vertical asymptote: Domain:

x-intercept: y-intercept: Hole:

$$f(x) = \frac{2x-6}{x-4} = \frac{2(x-3)}{x-4}$$

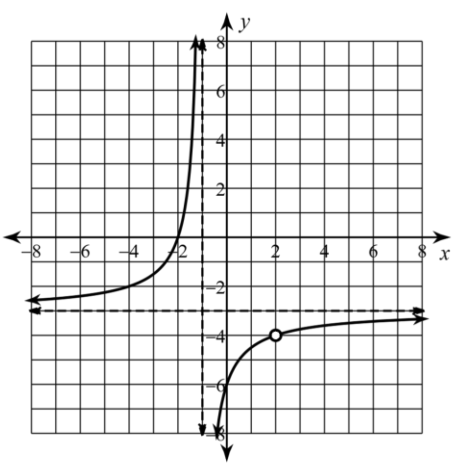


End behavior: $\lim_{x\to-\infty} f(x) =$ $\lim_{x\to\infty} f(x) =$

Horizontal asymptote: Vertical asymptote: Domain:

x-intercept: y-intercept: Hole:

$$f(x) = \frac{-3x^2+12}{x^2-x-2} = \frac{-3(x-2)(x+2)}{(x-2)(x+1)}$$



End behavior: $\lim_{x\to-\infty} f(x) =$ $\lim_{x\to\infty} f(x) =$

Horizontal asymptote: Vertical asymptote: Domain:

x-intercept: y-intercept: Hole:

Analysis & Summary:

How can you identify the following characteristics using the function formula? *Be sure to mention which function formula – standard form or factored form – you are using.*

- End behavior:
- Horizontal asymptote:
- Vertical asymptote:
- Domain:
- x-intercept:
- y-intercept:
- Hole:

5.3.D1 ~ The Behavior of Rational Functions

OBJECTIVES:

- Demonstrate an understanding of the characteristics, behaviors, formulas and graphs of rational functions
- Interpret a function describing a rational function given graphically, analytically, numerically, and verbally
- Analyze the long-run AND short-run behavior of rational function

❖ Rational Functions

➢ Rational functions are ratios (aka quotients) of polynomial functions: $r(x) = \frac{p(x)}{q(x)}$ where $p(x)$ & $q(x)$ are polynomial functions with $q(x) \neq 0$.

❖ THE LONG-RUN/END BEHAVIOR OF RATIONAL FUNCTIONS

➢ In the long run, every rational function behaves like a polynomial or a power function.

Refer to your summary of end behavior at the bottom of the previous page.

- Consider the following function:

$$f(x) = \frac{6x^4 + x^3 + 1}{-5x + 2x^2} \approx \frac{6x^4}{2x^2} = 3x^2$$

- The numerator behaves like $6x^4$ and the denominator behaves like $2x^2$.
  - Simplify this ratio.
- Therefore, the long-run behavior of *f* is ∞ because… $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} (3x^2) = \infty$

Examples:

For each rational function, find the indicated limit. *You may want to refer to the Leading Term Test.*

1. $\lim_{x\to-\infty} \frac{2 + 5x}{6x + 3}$

2. $\lim_{x\to-\infty} \frac{1 - 4x^2}{5x - 2}$

3. $\lim_{x\to\infty} \frac{3x^4 - 2}{5x^2}$

❖ HORIZONTAL ASYMPTOTES OF RATIONAL FUNCTIONS

➢ Compare the degree of the numerator, *N*, to the degree of the denominator, *D*:

Summary	*Compare degrees:*	$N° = D°$	$N° < D°$	$N° > D°$
"The Comparison Test"	**End Behavior**			
	Horizontal Asymptote			

Examples:

For each rational function, compare the degrees of the numerator and denominator and then find the equation of the horizontal asymptote, if there is one.

4. $f(x) = \frac{2x + 1}{x^2 - x}$

5. $k(x) = \frac{3x^2 + 2x - 1}{1 - 2x^2}$

6. $j(x) = \frac{x^3 + 1}{x + 2}$

❖ DOMAIN OF A RATIONAL FUNCTION

➢ The set of all real numbers <u>except</u> the x-values that make the denominator zero.

❖ HORIZONTAL & VERTICAL INTERCEPTS

➢ Horizontal/x-intercepts/zeros/roots
- These occur at zeros of the numerator, which are not also zeros of the denominator.

➢ Vertical/y-intercept
- This is the value of the function at $x = 0$, if defined.

"Dissect" this rational function:

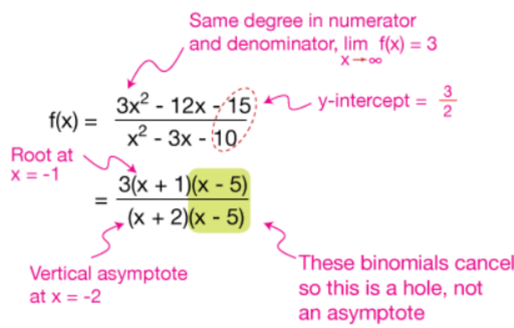
$$f(x) = \frac{2x-6}{x-4} = \frac{2(x-3)}{x-4}$$

❖ VERTICAL ASYMPTOTES OF RATIONAL FUNCTIONS

➢ A vertical asymptote of a function $f(x)$ is a vertical line, $x = a$, that the graph of $f(x)$ approaches but does not cross.
- These occur at zeros of the denominator, which are not also zeros of the numerator.
- *A rational function may have no vertical asymptotes, one vertical asymptotes, or several vertical asymptotes.*

❖ CAUTION: HOLES

➢ There is a hole corresponding to $x = a$, and **<u>not</u>** a vertical asymptote, in the graph of a rational function if the value a causes the denominator to be zero, but there is a reduced form of the function's equation in which a does not cause the denominator to be zero.
- Occur at zeros of the denominator also zeros of the numerator
  - *The zero of the common factor*
- To determine the y-coordinate of the hole, plug the x-value of the hole into the simplified form of the rational function.



❖ ANALYZING & GRAPHING RATIONAL FUNCTIONS

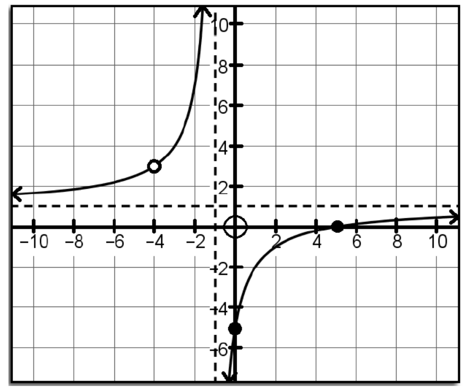
❶ End behavior asymptote/ horizontal asymptote	Compare the degrees of the numerator & denominator. If $N° < D°$, then $y = 0$. If $N° = D°$, then $y =$ ratio.
❷ y-intercept	Evaluate $f(0)$; plug 0 in for x & calculate.
❸ Factor the numerator and denominator	
❹ Identify the domain	Where does the denominator equal 0?
❺ Cancel out any common factors; write the "reduced function"	
❻ x-coordinate of hole	Zeros of the common factor(s)
y-coordinate of hole	Plug hole's x-coordinate into reduced function
❼ Vertical asymptote(s)	Zeros of the remaining factor(s) in the denominator
❽ x-intercept(s)	Zeros of the remaining factor(s) in the numerator

Example:

❶ factor and cancel; ❸ & ❺

$$f(x) = \frac{x^2 - x - 20}{x^2 + 5x + 4} \rightarrow \frac{\cancel{(x+4)}(x-5)}{\cancel{(x+4)}(x+1)} \rightarrow \text{"REDUCED FUNCTION"} \ \frac{(x-5)}{(x+1)}$$

❶ horizontal asymptote: $y = 1$
❷ y-intercept: $(0, -5)$ $\frac{(\mathbf{0} - 5)}{(\mathbf{0} + 1)} = \mathbf{-5}$
❹ Domain: $\{x \mid x \neq -4 \ \& -1\}$
❻ Hole: $(-4, 3)$ $\frac{(\mathbf{-4} - 5)}{(\mathbf{-4} + 1)} = \mathbf{3}$
❼ vertical asymptote: $x = -1$
❽ x-intercept: $(5, 0)$



This is the general shape of a rational function with one vertical asymptote.

Examples:

Use the function formula to identify the rational function's characteristics.

7.
$$f(x) = \frac{x^2 - 2x - 3}{x^2 + 5x + 4} = \frac{(x-3)(x+1)}{(x+1)(x+4)}$$

$\lim_{x \to -\infty} f(x) =$	$\lim_{x \to \infty} f(x) =$	Horizontal asymptote:	y-intercept:
Vertical asymptote:	x-intercept:	Hole:	Domain:

8.
$$f(x) = \frac{-2x + 2}{x^2 + 3x - 4} = \frac{-2(x-1)}{(x-1)(x+4)}$$

$\lim_{x \to -\infty} f(x) =$	$\lim_{x \to \infty} f(x) =$	Horizontal asymptote:	y-intercept:
Vertical asymptote:	x-intercept:	Hole:	Domain:

9.
$$f(x) = \frac{x^2 + 2x}{-4x + 8} = \frac{x(x+2)}{-4(x-2)}$$

$\lim_{x \to -\infty} f(x) =$	$\lim_{x \to \infty} f(x) =$	Horizontal asymptote:	y-intercept:
Vertical asymptote:	x-intercept:	Hole:	Domain:

5.3.D2 ~ Graphing Rational Functions

OBJECTIVES:

- Demonstrate an understanding of the characteristics, behaviors, formulas, and graphs of rational functions
- Analyze the long-run and the short-run behavior of rational functions
- Sketch the graph of a rational function

❖ Rational Functions

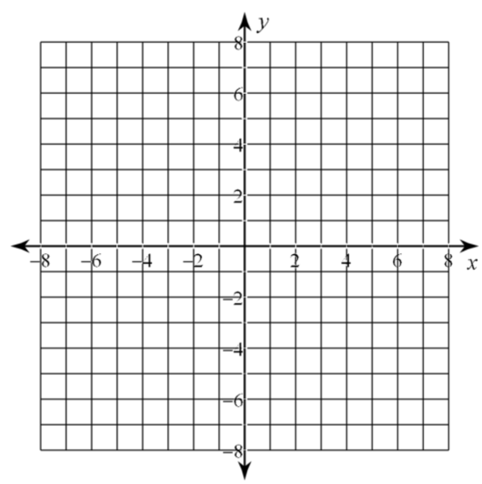
- Rational functions are ratios (aka quotients) of polynomial functions: $r(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ & $q(x)$ are polynomial functions with $q(x) \neq 0$.

Review the "Eight-Part Analysis" process on page 20.

Examples: Analyze the long-run and short run behavior of the rational function and then sketch its graph.

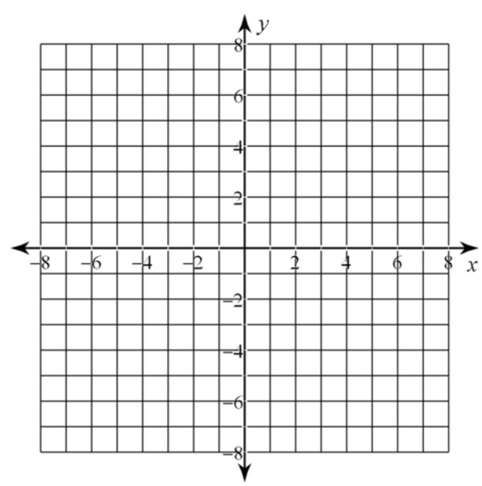
1. $f(x) = \dfrac{x^2 - 1}{-x^2 + x}$

$\lim_{x \to -\infty} f(x) =$	$\lim_{x \to \infty} f(x) =$	Horizontal asymptote:	y-intercept:
Vertical asymptote:	x-intercept:	Hole:	Domain:



2. $f(x) = \dfrac{3x^2 - 12}{x^2 - x - 6}$

$\lim_{x \to -\infty} f(x) =$	$\lim_{x \to \infty} f(x) =$	Horizontal asymptote:	y-intercept:
Vertical asymptote:	x-intercept:	Hole:	Domain:



5.3.D3 ~ Writing Rational Functions

OBJECTIVES:

- Demonstrate an understanding of the characteristics, behaviors, formulas, and graphs of rational functions
- Fit a rational function to a function given graphically, analytically, numerically, and verbally

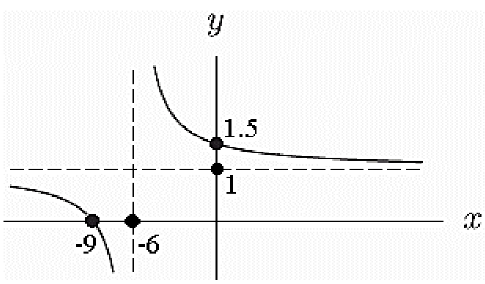
❖ Finding a Formula for a Rational Function from its Graph

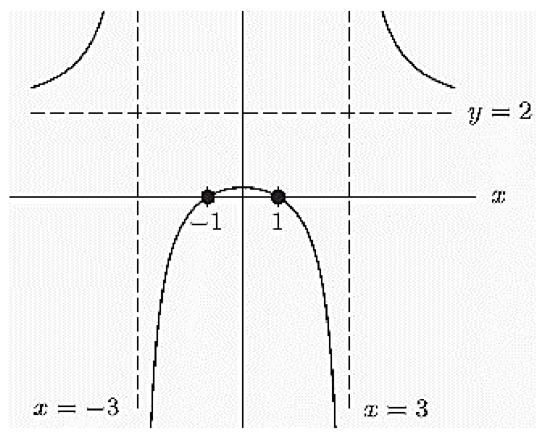
- ➢ From the graph of a rational function, we can find
  - The **FACTOR(S) OF THE NUMERATOR** from the **X-INTERCEPT(S)** of the function
    - *Pay attention to whether the graph crosses or simply touches the x-axis at each x-intercept – this could indicate any multiple zeros*
  - The **FACTOR(S) OF THE DENOMINATOR** from the **VERTICAL ASYMPTOTE(S)** of the function
  - Any **COEFFICIENT OF THE NUMERATOR** comes from the **HORIZONTAL ASYMPTOTE** of the function
    - *Remember: if the horizontal asymptote is 0, then $N° < D°$; if the horizontal asymptote is some other number, then $N° = D°$*
  - Any **FACTORS OF BOTH THE NUMERATOR AND THE DENOMINATOR** come from the **HOLE(S)** of the function.
  - The y-intercept tells you the value of the function when $x = 0$
    - *You can use the y-intercept to check your function.*

$$r(x) = \frac{\mathbf{H.A.}\,(x - x\mathbf{int})(x - \mathbf{hole})}{(x - \mathbf{V.A.})(x - \mathbf{hole})}$$

Examples:

Find a possible formula for the rational function graphed or described.

1. 

2. 

3. The graph has vertical asymptotes at x = 1 and x = 4, a horizontal asymptote at y = -3, and touches the x-axis at x = 2

4. The function has an x-intercept at $x = 1$, a hole at $x = -2$, a vertical asymptote at $x = -5$, and a horizontal asymptote at $y = 2$.