4. If $5000 is deposited in an account paying a nominal interest rate of 4% per year, how much is in the account 10 years later if interest is compounded...

$t=10$ $\quad 0.04$

a. Monthly? $n=12$

$$B=5000\left(1+\frac{0.04}{12}\right)^{120}$$

$7454.16

b. Continuously?

$$B=Pe^{rt}=5000e^{0.04\cdot 10}$$

$7459.12

5. A town has population 3000 people at year $t = 0$.
Determine whether the population can be expressed with...
   - a linear function $P = b + mt$
   - a non-continuous exponential function $P = a(b)^t$
   - a continuous exponential function $P = ae^{kt}$

Then write a formula for the population, P, in year t if the town...

a. Shrinks at a continuous rate of 4% per year. $P=3000e^{-0.04t}$

b. Grows by 200 people per year $P=3000+200t$

c. Minimizes by 50 people every 52 weeks. $P=3000-50t$

d. Escalates at a continuous rate of 6% annually. $P=3000e^{0.06t}$

e. Diminishes by 4% every 365 days. $P=3000(0.96)^t$

f. Boosts by 6% every 12 months. $P=3000(1.06)^t$

6.4.D1 ~ Logarithms & Exponents

OBJECTIVES
- Demonstrate an understanding of logarithms & their properties
- Convert from logarithmic form to exponential form (and vice versa)
- Evaluate & simplify expressions with logarithms
- Solve exponential equations using properties of exponents

❖ Logarithms
➢ An exponential function, $f(x) = b^x$, has an inverse that is a function. This inverse is the logarithmic function with base b, denoted $\log_b x$.

❖ Converting Between Logarithmic & Exponential Form

$$y = \log_b x \leftrightarrow b^y = x$$

Examples:
Write each equation in its equivalent exponential form.

1. $\log_b 64 = 3$ $\quad b^3=64$

2. $y = \log_3 7$ $\quad 3^y=7$

3. $\log_5 x = 2$ $\quad 5^2=x$

- A logarithm is an EXPONENT
  - We can evaluate simple logarithmic expressions using our understanding of exponents.

- **One-to-One Property of Exponents**
  - If $b^M = b^N$, then $M = N$.
    - Express each side of the equation as a power of the same base.
    - Set the exponents equal to each other & then solve for the variable.

- **Properties of Exponents**
  - Let a, b, x, and y be real numbers with a & $b > 0$,

$b^0 = 1$ $\qquad b^{-x} = \frac{1}{b^x}$ $\qquad (a^m)^n = a^{mn}$ $\qquad \sqrt{x} = x^{\frac{1}{2}}$ $\qquad \sqrt[n]{x^m} = x^{\frac{m}{n}}$

Logarithmic Expression	Question Needed for Evaluation	Logarithmic Expression Evaluated
$16 = 2^4$ **a.** $\log_2 16$	2 to what power gives 16? $2^? = 16$ $\quad 2^x = 2^4 \quad x = 4$	$\log_2 16 = 4$ because $2^4 = 16$.
$\frac{1}{49} = 7^{-2}$ **b.** $\log_7 \frac{1}{49}$	7 to what power gives $\frac{1}{49}$? $7^? = \frac{1}{49}$ $\quad 7^x = 7^{-2} \quad x = -2$	$\log_7 \frac{1}{49} = -2$ because $7^{-2} = \frac{1}{7^2} = \frac{1}{49}$.
$5 = 25^{1/2}$ **c.** $\log_{25} 5$	25 to what power gives 5? $25^? = 5$ $\quad 25^{1/2} = 5^x$	$\log_{25} 5 = \frac{1}{2}$ because $25^{\frac{1}{2}} = \sqrt{25} = 5$.
$\sqrt[5]{2} = 2^{1/5}$ **d.** $\log_2 \sqrt[5]{2}$	2 to what power gives $\sqrt[5]{2}$, or $2^{\frac{1}{5}}$? $2^? = \sqrt[5]{2}$	$\log_2 \sqrt[5]{2} = \frac{1}{5}$ because $2^{\frac{1}{5}} = \sqrt[5]{2}$.

<u>Examples</u>:

Evaluate the logarithm by converting it to exponential form. Use the properties of exponents to simplify the logarithm.

4. $\log_3 \sqrt{3} = y$

$3^y = \sqrt{3}$ $\qquad \sqrt{x} = x^{1/2}$

$3^y = 3^{1/2}$

$y = 1/2$

5. $\log_5 \frac{1}{25} = y$

$5^y = \frac{1}{25} = \frac{1}{5^2}$ $\qquad \frac{1}{b^x} = b^{-x}$

$5^y = 5^{-2}$

$y = -2$

Use the One-to-One Properties of Exponents to solve the exponential equation.

6. $27^{-2x} = \frac{1}{81} = \frac{1}{3^4}$

$(3^3)^{-2x} = 3^{-4}$

$3(-2x) = -4$

$\frac{-6x}{-6} = \frac{-4}{-6}$

$x = 2/3$

7. $64^{3-x} = 4^{x-3}$

$(4^3)^{3-x} = 4^{x-3}$

$3(3-x) = x-3$

$9 - 3x = x - 3$

$12 = 4x$

$3 = x$

6.4.D2 ~ Solving Logarithmic Equations

OBJECTIVES

- Demonstrate an understanding of logarithms & their properties
- Solve logarithmic equations using properties of logarithms and the definition of a logarithm

❖ What is a logarithm?

- The inverse function of the exponential function with base b is called the logarithmic function with base b: $f(x) = \log_b x$.
  - For $x > 0$ & $0 < b \neq 1$, then $y = \log_b x$ is equivalent to $b^y = x$.

❖ Using the Definition of a Logarithm to Solve Logarithmic Equations

- Express the equation in the form: $\log_b M = c$ (*Get the logarithm alone.*)
- Use the definition of a logarithm to rewrite the equation in exponential form: $b^c = M$
- Solve for the variable.
- Check for extraneous solutions: $M > 0$.
  - Exclude from the solution set any proposed solution that produces the logarithm of a negative number or the logarithm of 0.

1. $\frac{2\log_2(4x+1)}{2} = \frac{10}{2}$

$\log_2(4x+1) = 5$

$2^5 = 4x+1$

$32 = 4x+1$

$31 = 4x$

$7.75 = x$

check! $4(7.75)+1 > 0$ ok

❖ Using the One-to-One Property of Logarithms to Solve Logarithmic Equations

- Express the equation in the form: $\log_b M = \log_b N$ done
- Use the one-to-one property to rewrite the equation without logarithms:
  - If $\log_b M = \log_b N$, then $M = N$.
- Solve for the variable.
- Check for extraneous solutions: $M > 0$ & $N > 0$

2. $\log_6(4x+9) = \log_6 21$

$4x+9 = 21$

$4x = 12$

$x = 3$

$4(3)+9 > 0$ ok

IT IS SOMETIMES NECESSARY TO USE PROPERTIES OF LOGARITHMS TO CONDENSE LOGARITHMS INTO A SINGLE LOGARITHM.

❖ Properties of Logarithms

- Let b, R, and S be positive real numbers with $b \neq 1$, and c any real number,
  - (2) Product rule: $\log_b R + \log_b S = \log_b(RS)$
  - Quotient rule: $\log_b R - \log_b S = \log_b\left(\frac{R}{S}\right)$
  - (1) Power rule: $\log_b(R^c) = c\log_b R$

Examples: Use the properties of logarithms to condense the expression.

3. $5\log_9 a + \log_9 b$

1) Power Rule $\log_9 a^5 + \log_9 b$

(2) Product Rule $\log_9(a^5 b)$

4. $6\log_8 x - 2\log_8 y$

Power Rule

(1) $\log_8 x^6 - \log_8 y^2$

(2) $\log_8\left(\frac{x^6}{y^2}\right)$ Quotient Rule

Examples:

Use a property of logarithms to condense the left side of the equation. Then use the appropriate method to solve the logarithmic equation. *If necessary, round to two decimal places. Remember to check for extraneous solutions.*

5. $\log_9 8 + \log_9(x-10) = \log_9 26$

Product Rule (1) $\log_9(8(x-10)) = \log_9 26$

One to one prop. (2) $8x - 80 = 26$

Log M = Log N = M=N

$8x = 106$

$x = 13.25$

6. $\log_2(x+6) - \log_2 x = 5$ (1) Quotient Rule

$\log_2\left(\frac{x+6}{x}\right) = 5$

(2) Def. of log

$2^5 = \frac{x+6}{x}$

$32 = \frac{x+6}{x}$

$32x = x + 6$

$31x = 6$

$x = \frac{6}{31}$

6.4.D3 ~ Solving Exponential Equations

OBJECTIVE

- Solve exponential equations using logarithms

Recall: Logarithms "undo" exponentials, so logs are used to solve exponential equations.

MOST EXPONENTIAL EQUATIONS CANNOT BE REWRITTEN SO THAT EACH SIDE HAS THE SAME BASE.

❖ **Using Logarithms to Solve Exponential Equations**
- Isolate the exponential expression.
- Take the natural logarithm of both sides.
- Simplify using one of the following properties:
  - $\ln b^x = x \ln b$
  - $\ln e^x = x$
- Solve for the variable.

$4^x = 15$

We cannot rewrite both sides in terms of base 2 or base 4.

$\ln(4^x) = \ln 15$

$x \cdot \ln 4 = \ln 15$

$x = \frac{\ln 15}{\ln 4}$

$x \approx 1.953$

Examples:

Solve the equation using a logarithm. Provide an exact answer, solve for *x*, and an approximate answer, rounded to three decimal places.

1. $\frac{15(2.3)^x}{15} = \frac{63}{15}$ (1) Isolate

$(2.3)^x = 4.2$ (2) Ln & simplify

$\frac{x \cdot \cancel{\ln 2.3}}{\cancel{\ln 2.3}} = \frac{\ln 4.2}{\ln 2.3} \approx 1.723$

2. $7^{x+2} = 410$ (1) Ln & simplify

$\frac{(x+2)\ln 7}{\ln 7} = \frac{\ln 410}{\ln 7}$

(2) Solve for the variable

$x + 2 - 2 = \frac{\ln 410}{\ln 7} - 2$

$x \approx 1.092$

3. $\frac{10e^{4x+1}}{10} = \frac{20}{10}$ **Recall: $\ln e^x = x$**

(1) Isolate

(2) Ln & simplify

$e^{4x+1} = 2$

$\frac{4x+\cancel{1}-1}{4} = \frac{\ln 2 - 1}{4} \approx -0.077$

6.5.D1 ~ Logarithms & Exponential Models

OBJECTIVES

- Analyze logarithms & exponential models
- Set up exponential equations given verbally
- Solve applied problems involving logarithmic functions

❖ Doubling Time

➢ Eventually any exponentially growing quantity doubles, or increases by 100%. The time it takes a quantity to do this is known as doubling time.

- If $Q = a(1+r)^t$, the initial quantity, a, doubles, therefore: $Q = 2a$.
- Then $2a = a(1+r)^t$, or simply: $2 = (1+r)^t$.

	Non-Continuous		Continuous
Doubling Time	$2 = b^t$	$2 = (1+r)^t$	$2 = e^{kt}$

Example:

1. In 2005, Sudan's population was 40.19 million and has been increasing by 2.6% every year. Estimate the number of years it will take for the population to double.

$b = 1 + R$

$b = 1.026$

$2 = 1.026^t$

use Ln to solve

$\frac{\ln 2}{\ln 1.026} = \frac{t \cdot \cancel{\ln 1.026}}{\cancel{\ln 1.026}}$

$t \approx 27$ years

❖ Half Life

➢ The half-life of a substance is the amount of time it takes for half of the substance to decay.

- If $Q = a(1-r)^t$, the initial quantity, a, halves, therefore: $Q = {}^1/_2\, a$.
- Then ${}^1/_2\, a = a(1-r)^t$, or simply: ${}^1/_2 = (1-r)^t$.

	Non-Continuous		Continuous
Half-Life	$\frac{1}{2} = b^t$	$\frac{1}{2} = (1-r)^t$	$\frac{1}{2} = e^{-kt}$

Examples:

2. Tritium decays at a continuous rate of 5.471%. Find the half-life of Tritium.

$\frac{1}{2} = e^{-0.05471t}$

$\ln 1/2 = -0.05471t$

$t \approx 12.67$ years

3. The half-life of a Twinkie is 14 days. (I think this is after it has been removed from its plastic packaging because, according to *Zombieland,* Twinkies will survive the apocalypse.) Find the daily decay rate of the Twinkie.

$\frac{1}{2} = b^{14} \longrightarrow \left(\frac{1}{2}\right)^{1/14} = \sqrt[14]{b^{14}}$

not an expo. function

$b \approx 0.9517$

Rate = 4.83%

6.5.D2 ~ Logarithms & Exponential Models

$b=e^k$

$a(b)^t$ $\quad$ ae^{kt}

$K=\ln b$

OBJECTIVES:

- Analyze logarithms & exponential models
- Set up exponential equations given verbally & solve using logarithms
- Rewrite any exponential in the form $Q = ae^{kt}$

❖ **Converting Between $Q = a(b)^t$ & $Q = ae^{kt}$**

	Non-Continuous	Continuous
Function Formula	$Q = a(b)^t$	$Q = ae^{kt}$
Convert	$b = e^k$	$k = \ln b$
Rate Rule	$(b-1) \times 100\%$	$k \times 100\%$

- If k is positive, then Q is a continuous growth function
- If k is negative, then Q is a continuous decay function

Examples:

1. Fill in the table below, assuming that t is measured in years. Round k and b values to 4 decimal places.

$b=e^{-0.04}$

$K=\ln 1.2$

$K=\ln 0.91$

Formula: $Q = a(b)^t$	Formula: $Q = ae^{kt}$	Growth or Decay?	Rate	Continuous Rate
$Q=6(0.9608)^t$	$Q = 6e^{-0.04t}$	decay	3.92%	4%
$Q = 5(1.2)^t$	$Q=5e^{0.1823t}$	growth	20%	18.23%
$Q = 10(0.91)^t$	$Q=10e^{-0.0943t}$	decay	9%	9.43%

2. The population of a bacteria colony starts at 100 and grows by 30% per hour.
   a. Find an exponential function model for the number of bacteria, B, after t hours.

   $B = 100(1.30)^t$

   b. What is the continuous growth rate of the colony?

   $K \times 100\%$
   $\ln b \times 100\%$ $\qquad$ $K=\ln 1.30 \times 100 = 26.236\%$

3. A population doubles in size every 20 years. Assuming exponential growth, find the...
   a. Annual growth rate.

   3.53% $\qquad$ $2=b^{20} \rightarrow 2^{1/20}=(b^{20})^{1/20}$
   $1.0353=b$

   b. Continuous growth rate.

   $K=\ln b \times 100\%$
   $K=\ln 1.0353 \times 100 \approx 3.469\%$

4. The population of Spring Grove grew from 11,000 to 13,000 in three years.

0, 3,

   a. The population of Spring Grove, can be represented by an exponential function. Calculate the change factor. *Round the change factor to 3 decimal places.*

$$b = \left(\frac{y_2}{y_1}\right)^{\frac{1}{x_2 - x_1}}$$

$b = \left(\frac{13000}{11000}\right)^{1/3} \approx 1.057$

   b. What is the annual growth rate? 5.7%

   c. What is the continuous growth rate?

$K = \ln b \times 100\%$

$\ln 1.057 \times 100\% \approx 5.543\%$

Backer Breaks it Down: Non-Continuous vs. Continuous

	Non-Continuous		Continuous
Function Formula	$Q = a(b)^t$		$Q = ae^{kt}$
Growth	$b = 1 + r$		$k = +r$
Decay	$b = 1 - r$		$k = -r$
Compound Interest	$B = P\left(1 + \frac{r}{n}\right)^{nt}$		$B = Pe^{rt}$
Effective Rate	$APY = \left[\left(1 + \frac{r}{n}\right)^n - 1\right] \times 100\%$		$APY = [e^r - 1] \times 100\%$
Convert	$b = e^k$		$k = \ln b$
Rate Rule	$(b - 1) \times 100\%$		$k \times 100\%$
Doubling Time	$2 = b^t$	$2 = (1 + r)^t$	$2 = e^{kt}$
Half-Life	$\frac{1}{2} = b^t$	$\frac{1}{2} = (1 - r)^t$	$\frac{1}{2} = e^{-kt}$
Other	$b = \left(\frac{y_2}{y_1}\right)^{\frac{1}{x_2 - x_1}}$		