CHAPTER 3 ~ DERIVATIVES

DEFINITION **Derivative**

The **derivative** of the function f with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \qquad (1)$$

provided the limit exists.

Notation

There are many ways to denote the derivative of a function $y = f(x)$. Besides $f'(x)$, the most common notations are these:

y'	"y prime"	Nice and brief, but does not name the independent variable.
$\frac{dy}{dx}$	"dy dx" or "the derivative of y with respect to x"	Names both variables and uses d for derivative.
$\frac{df}{dx}$	"df dx" or "the derivative of f with respect to x"	Emphasizes the function's name.
$\frac{d}{dx}f(x)$	"d dx of f at x" or "the derivative of f at x"	Emphasizes the idea that differentiation is an operation performed on f.

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3.2 DIFFERENTIATION RULES

OBJECTIVE: USE DIFFERENTIATION RULES TO FIND THE DERIVATIVE OF A FUNCTION ANALYTICALLY

Integer Powers, Multiples, Sums, and Differences

- Derivative of a Constant Function
  - If f is the function with the constant value c, then…

$$\frac{df}{dx} = \frac{d}{dx}(c) = \_\_\_\_\_\_\_\_$$

- Power Rule for Integer Powers of x
  - If n is an integer, then…

$$\frac{d}{dx}(x^n) = \_\_\_\_\_\_\_\_\_\_\_$$

Recall that: $\frac{\mathbf{1}}{\boldsymbol{x^m}} = \boldsymbol{x^{-m}}$

    - To differentiate x^n, multiply by n and subtract 1 from the exponent.
- The Constant Multiple Rule
  - If u is a differentiable function of x and c is a constant, then…

$$\frac{d}{dx}(cu) = \_\_\_\_\_\_\_\_\_\_\_$$

    - If a differentiable function is multiplied by a constant, then its derivative is multiplied by the same constant.

TO FIND THE DERIVATIVES OF POLYNOMIALS, WE NEED TO BE ABLE TO DIFFERENTIATE SUMS AND DIFFERENCES OF MONOMIALS.

- The Sum & Difference Rule
  - If u and v are differentiable functions of x, then their sum and difference are differentiable at every point where u and v are differentiable. At such points…

$$\frac{d}{dx}(u \pm v) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Examples: Finding dy/dx

1. $y = \sqrt{3}$
2. $y = x^3$
3. $y = 3x^4$
4. $y = \frac{5x^3}{3}$
5. $y = \frac{4}{x^3}$
6. $y = \frac{7x^{-2}}{4}$
7. $y = x^4 + 12x^{-1}$
8. $y = x^3 + 3x^2 - 5x + 1$

Example 9: Application of Derivative ~ Horizontal Tangents

Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

The horizontal tangents, if any, occur where the slope dy/dx is zero.

(a) Calculate dy/dx

(b) Solve $dy/dx = 0$ for x

Products & Quotients

- The Product Rule
  - The product of two differentiable functions u and v is differentiable, and...

$$\frac{d}{dx}(uv) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

SCHULTZ SAYS:

"THE FIRST TIMES THE DERIVATIVE OF THE SECOND; PLUS THE SECOND TIMES THE DERIVATIVE OF THE FIRST."

  - You could also expand the original expression and then differentiate the resulting polynomial.

Example: The Product Rule

10. Find $f'(x)$ if $f(x) = (x^2 + 1)(x^3 + 3)$.

Via the Product Rule:

Multiply, then differentiate:

Product & Quotients (continued)

- The Quotient Rule
  - At a point where $v \neq 0$, the quotient $y = u/v$ of two differentiable functions is differentiable and...

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

SCHULTZ SAYS:

"THE DENOMINATOR TIMES THE DERIVATIVE OF THE NUMERATOR; MINUS THE NUMERATOR TIMES THE DERIVATIVE OF THE DENOMINATOR; ALL OVER THE DENOMINATOR SQUARED."

Examples: The Quotient Rule

11. Differentiate $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$

Tangent Lines

- When $f'(x)$ exists for a particular x, it is called the slope of the curve $y = f(x)$ at x.
  - The line that passes through the point $P(x, f(x))$ with slope $f'(x)$ is the tangent to the curve at P.

Example 12: Application of Derivative ~ Tangent to a Curve

Find an equation for the tangent to the curve at the point $(1, 3)$.

$$y = x + \frac{2}{x}$$

(a) Find the slope of the curve:

(b) Find the slope at $x = 1$:

(c) Find the line through $(1, 3)$ w/the slope found in (b)

Second & Higher Order Derivatives

First derivative	Second derivative	Third derivative	nth derivative
$y' = \dfrac{dy}{dx}$	$y'' = \dfrac{dy'}{dx} = \dfrac{d^2y}{dx^2}$	$y''' = \dfrac{dy''}{dx}$	$y^{(n)} = \dfrac{d}{dx} y^{(n-1)}$

Example: Higher-Order Derivatives

14. Find the first four derivatives of $y = x^3 - 3x^2 + 2$.

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3.3 RATES OF CHANGE

OBJECTIVE: USE DERIVATIVES TO REPRESENT THE RATES AT WHICH THINGS CHANGE

- Free Fall
  - The equation for the distance an object falls freely from rest is…

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    - s is distance, t is time, and g is the constant acceleration given to an object by the force of gravity
      - $g = 32 \text{ ft/sec}^2$
      - $g = 9.80 \text{ m/sec}^2$

Example 1: Free Fall

A heavy ball bearing is released from rest at time $t = 0$.

a. How many feet does the ball bearing fall in the first 2 seconds?

b. How long does it take the ball bearing to fall the first 14.7 meters?

- Motion Along a Line

  Suppose that an object is moving along a coordinate line so that we know its position s on that line as a function of time t: $s = f(t)$
  - Displacement – the object's net change in position:

$$\Delta s = f(t + \Delta t) - f(t)$$

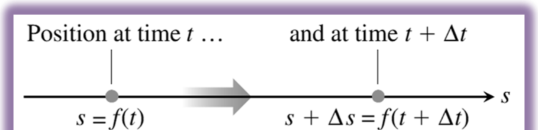

Figure 3.23 The positions of an object moving along a coordinate line at time t and shortly later at time $t + \Delta t$.
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- The <u>average velocity</u> of the object is Δs (change in position) divided by Δt (change in time)

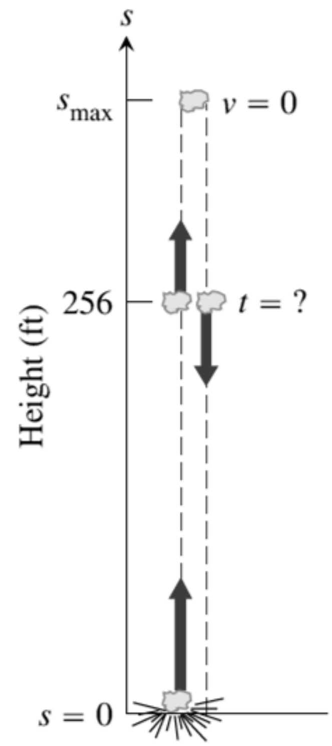
$$\mathrm{v_{av}} = \frac{\text{displacement}}{\text{travel time}} = \frac{\Delta s}{\Delta t} = \frac{f(t-\Delta t)-f(t)}{\Delta t}$$

- The object's velocity at the exact instant t – aka <u>instantaneous velocity</u> (velocity) – is the derivative of the position function $s = f(t)$ with respect to time:

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t-\Delta t)-f(t)}{\Delta t}$$

<u>Example 2: Velocity</u>

A dynamite blast propels a heavy rock straight up with a launch velocity of 160 ft/sec (about 109 mph). It reaches a height of $s = 160t - 16t^2$ feet after t seconds.



a. How high does the rock go?

   The rock reaches its maximum height at the position where its upward velocity is zero.

b. How fast is the rock traveling when it is 256 feet above the ground on the way up? On the way down?

Motion Along a Line (continued)

- <u>Speed</u> – the magnitude of velocity

$$\text{speed} = |v(t)| = \left|\frac{ds}{dt}\right|$$

- <u>Acceleration</u> – the rate at which an object's velocity changes
  - Measures how quickly the body picks up or loses speed

$$\text{acceleration} = a(t) = \frac{dv}{dt} = \frac{d^2v}{dt^2}$$

Return to Example 2

c. What is the acceleration of the rock at any time t during its flight (after the blast)?

d. When does the rock hit the ground?

The rock hits the ground at the positive time for which $s = 0$.

Derivatives in Economics

- The cost of production $c(x)$ is a function of x, the number of units produced
- The marginal cost of production dc/dx is the rate of change of cost with respect to the level of product

Examples: Derivatives in Economics

3. Suppose it costs $c(x) = x^3 - 6x^2 + 15x$ dollars to produce x stoves and your shop is currently producing 10 stoves a day. About how much extra will it cost to produce one more stove a day?

4. If $r(x) = x^3 - 3x^2 + 12x$ gives the dollar revenue from selling x thousand candy bars.
   a. Find the marginal revenue when x thousand candy bars are sold.
   b. If you currently sell 10 thousand candy bars a week, what can you expect your revenue to increase by if you increase sales to 11 thousand bars a week?

3.4 DERIVATIVES OF TRIG FUNCTIONS

OBJECTIVE: USE DIFFERENTIATION RULES TO FIND THE DERIVATIVE OF EXPRESSIONS INVOLVING TRIGONOMETRIC FUNCTIONS

Derivatives of Sine & Cosine

$$\frac{d}{dx}\sin x = \underline{\hspace{3cm}} \qquad \frac{d}{dx}\cos x = \underline{\hspace{3cm}}$$

Examples: Finding dy/dx

1. $y = 5x + \cos x$

2. $y = x^2 - \sin x$

3. $y = \sin x \cos x$

4. $y = \dfrac{\sin x}{x}$

5. $y = \dfrac{\cos x}{1 - \sin x}$

Derivatives of the Other Basic Functions

- Because $\sin x$ & $\cos x$ are differentiable functions of x, the related functions are differentiable at every value of x at which they are defined.

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x}$$

- Their derivatives are calculated from the quotient rule.

$$\frac{d}{dx}(\tan x) = \_\_\_\_\_\_\_\_\_\_ \qquad \frac{d}{dx}(\cot x) = \_\_\_\_\_\_\_\_\_\_$$

$$\frac{d}{dx}(\sec x) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_ \qquad \frac{d}{dx}(\csc x) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Examples: Finding dy/dx

6. $y = 1 + x - \cos x$

7. $y = 2\sin x - \tan x$

8. $y = x \sec x$

9. $y = 4 - x^2 \csc x$

10. $y = \dfrac{x}{1 + \cot x}$

Simple Harmonic Motion

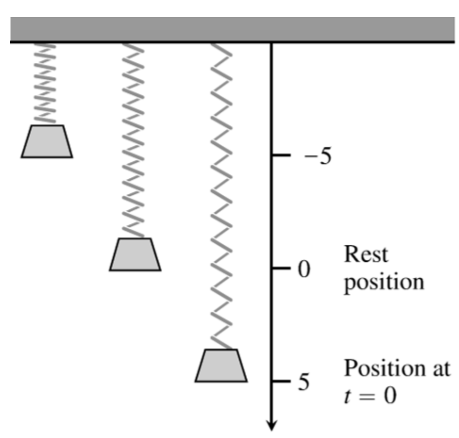
- The motion of a body bobbing up and down on the end of a spring is an example of simple harmonic motion.

Example 11: Simple Harmonic Motion

A body hanging from a spring is stretched 5 units beyond its rest position and released at time $t = 0$ to bob up and down. Its position at any later time t is:

$$s = 5\cos t$$

What are its velocity and acceleration at time t?



Jerk

- A sudden change in acceleration is called a "jerk."
- Jerk is the derivative of acceleration.
  - If a body's position at time t is $s = f(t)$, the body's jerk at time t is:

$$j = \frac{da}{dt} = \frac{d^3s}{dt^3}$$

<u>Return to Example 11</u>

What is the jerk of the simple harmonic motion in Example 11?

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3.5 CHAIN RULE

OBJECTIVE: USE THE CHAIN RULE TO DERIVE COMPOSITE FUNCTIONS

We know how to differentiate $\sin x$ and $x^2 - 4$, but how do we differentiate a composite like $\sin(x^2 - 4)$?

The Chain Rule

- If f is differentiable at the point $u = g(x)$, and g is differentiable at x, then the compositie function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and...

  ____

- Leibniz Notation:

  If $y = f(u)$ and $u = g(x)$, then...

$$\frac{dy}{dx} = \underline{\hspace{6cm}}$$

  ...where dy/du is evaluated at $u = g(x)$.

"Outside-Inside" Rule

- It sometimes helps to think about the Chain Rule this way: If $y = f(g(x))$, then...

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$$

  In words, differentiate the "outside" function f and evaluate it at the "inside" function $g(x)$ left alone; multiply by the derivative of the "inside" function.

$$\frac{d}{dx}\sin\underbrace{(x^2 + x)}_{\text{inside}} = \cos\underbrace{(x^2 + x)}_{\text{inside left alone}} \cdot \underbrace{(2x + 1)}_{\text{derivative of the inside}}$$
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Integer Powers of Differentiable Functions

The Chain Rule enables us to differentiate powers like $y = \sin^5 x$ & $y = (2x+1)^{-3}$ because these powers are composites:

$y = \sin^5 x$	is	u^5	with	$u = \sin x$
$y = (2x+1)^{-3}$	is	u^{-3}	with	$u = 2x + 1$

$\sin^5 x$ is short for $(\sin x)^5$

- Power Chain Rule
  - If u^n is an integer power of a differentiable function $u(x)$, then u^n is differentiable and…

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$$

Examples: Differentiating w/the Chain Rule

1. $y = \cos(1 - 2x)$

2. $y = \cot\frac{2}{x}$

3. $y = (x^2 + x + 1)^{-3}$

4. $y = \frac{1}{\sin^2 x} - \frac{2}{\sin x}$

5. $y = 2\tan^2 x - \sec^2 x$

Use the Chain Rule in combination with other differentiation rules to find the derivative of the function with respect to x.

6. $y = \sec^7(7x + 7)$

7. $y = x^{-3}(2x - 5)^4$

8. $y = \left(\frac{\sin x}{\cos x - 1}\right)^2$

9. Find the value of $(f \circ g)'$ at the given value of x.

$$f(u) = u^5 + 1, \qquad u = g(x) = \sqrt{x}, \qquad x = 1$$

3.6 Implicit Differentiation & Rational Powers

OBJECTIVE: USE IMPLICIT DIFFERENTIATION TO DERIVE FUNCTIONS THAT ARE NOT DEFINED OR WRITTEN EXPLICITLY AS A FUNCTION OF A SINGLE VARIABLE

Solve the equation for y: $x^3 + y^3 - 9xy = 0$

- Implicit Differentiation
  - The process by which we find dy/dx is called <u>implicit differentiation</u>.
    1.
    2.
    3.
    4.

<u>Examples: Using Implicit Differentiation</u>

1. $xy + 2x + 3y = 1$

2. $x^2 + xy + y^2 - 5x = 2$

3. $2y = x^2 + \sin y$

Implicit differentiation can also produce derivatives of higher order.

4. Find $\frac{d^2y}{dx^2}$ if $2x^3 - 3y^2 = 7$

Rational Powers of Differentiable Functions

- Power Rule for Rational Powers of x
  - If n is any rational number, then...

$$\frac{d}{dx}x^n = nx^{n-1}$$

  - If $n < 1$, then the derivative does not exist at $x = 0$.

Recall that:

$$\sqrt[n]{x^m} = x^{\frac{m}{n}}$$

Examples: Find dy/dx

5. $y = x^{\frac{9}{4}}$

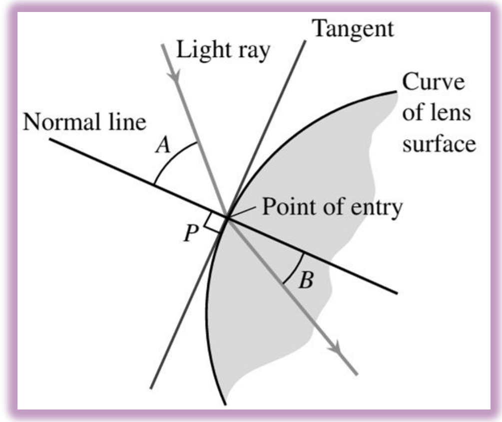
6. $y = \sqrt[3]{2x}$

7. $y = 7\sqrt{x+6}$

8. $x(x^2+1)^{1/2}$

Lenses, Tangent, and Normal Lines

- A line is normal to a curve at a point if it is perpendicular to the curve's tangent there.
  - The line is called the normal to the curve at that point.



Example 9: Tangent & Normal Lines

Find the tangent and the normal to the curve $x^2 - xy + y^2 = 7$ at the point $(-1, 2)$.