

Definitions:

- **Exponential function:** For  $a > 0$ , the exponential function with base  $a$  is defined by

$$f(x) = a^x$$

- **Horizontal asymptote:** The line  $y = c$  is a horizontal asymptote of the function  $f$  if

$$f(x) \rightarrow c \quad \text{as} \quad x \rightarrow \infty \quad \text{or} \quad x \rightarrow -\infty.$$

Properties of the graph of  $f(x) = a^x$ ,  $a > 0$ 

- Domain is all real numbers.
- Range is  $(0, \infty)$ .
- Always crosses through the point  $(0, 1)$ .
- $y = 0$  is a horizontal asymptote.
- If  $a > 1$ , then the function is increasing; if  $0 < a < 1$ , then the function is decreasing.

Important Formulas: Exponential functions are used in a variety of important formulas.

- **Compound Interest:** is calculated by the formula

$$A(t) = P \left( 1 + \frac{r}{n} \right)^{nt}$$

where

$A(t)$  = amount after  $t$  years

$P$  = principal

$r$  = interest rate

$n$  = number of times the interest is compounded per year

$t$  = number of years

- **Compounded Continuously Interest:** is calculated by the formula

$$A(t) = Pe^{rt}$$

where

$A(t)$  = amount after  $t$  years

$P$  = principal

$r$  = interest rate

$t$  = number of years

- **Exponential Growth:** of a population increases according to the formula

$$P(t) = P_0 e^{rt}$$

where

$P(t)$  = population after time  $t$

$P_0$  = initial population

$r$  = growth rate

$t$  = time

**Important Properties:**

- Every exponential function is a one-to-one function and hence has an inverse.

**Common Mistakes to Avoid:**

- Do NOT use the compounded continuously formula unless it says *compounded continuously* in the problem.
- In the exponential growth and compounded continuously formulas the  $rt$  is the exponent on  $e$ . Do NOT multiply  $e$  by  $rt$ .
- Remember to convert all interest or growth rates to a decimal before substituting into a formula.

## PROBLEMS

1. **If \$5,000 is invested at a rate of 8%, compounded weekly, find the value of the investment after 7 years.**

Here  $P = 5000$ ,  $t = 7$ ,  $r = .08$  and  $n = 52$  since there are 52 weeks in a year. Substituting these values into our compound interest formula, we get

$$\begin{aligned} A(t) &= P \left(1 + \frac{r}{n}\right)^{nt} \\ A(7) &= 5000 \left(1 + \frac{.08}{52}\right)^{52 \cdot 7} \\ &= 5000 (1.001538462)^{364} \\ &= 8749.596496 \end{aligned}$$

\$8749.60

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2. **If \$4000 is borrowed at a rate of 16% interest per year, compounded quarterly, find the amount due at the end of 4 years? 8 years?**

Here  $P = 4000$ ,  $r = .16$  and  $n = 4$  since there are 4 quarters in a year. To find the amount due at the end of 4 years we let  $t = 4$ . Substituting into the compound interest formula, we get

$$\begin{aligned} A(t) &= P \left(1 + \frac{r}{n}\right)^{nt} \\ A(4) &= 4000 \left(1 + \frac{.16}{4}\right)^{4 \cdot 4} \\ &= 4000 (1.04)^{16} \\ &= 7491.924983 \end{aligned}$$

\$7491.92 due at the end of 4 years

To find the amount due at the end of 8 years, we change  $t = 8$ .

$$\begin{aligned} A(t) &= P \left(1 + \frac{r}{n}\right)^{nt} \\ A(8) &= 4000 \left(1 + \frac{.16}{4}\right)^{4 \cdot 8} \\ &= 4000 (1.04)^{32} \\ &= 14032.23499 \end{aligned}$$

\$14032.23 due at the end of 8 years

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3. **If \$3000 is borrowed at a rate of 12% interest per year, find the amount due at the end of 5 years if the interest is compounded annually? monthly? daily?**

For this problem, we have  $P = 3000$ ,  $r = .12$  and  $t = 5$ .

If the money is compounded annually, then  $n = 1$ . Substituting into the compound interest formula, we get

$$\begin{aligned} A(t) &= P \left(1 + \frac{r}{n}\right)^{nt} \\ A(5) &= 3000 \left(1 + \frac{.12}{1}\right)^{1 \cdot 5} \\ &= 3000 (1.12)^5 \\ &= 5287.02505 \end{aligned}$$

\$5287.03 due if compounded annually

When the money is compounded monthly,  $n = 12$  since there are 12 months in one year. Therefore,

$$\begin{aligned} A(t) &= P \left(1 + \frac{r}{n}\right)^{nt} \\ A(5) &= 3000 \left(1 + \frac{.12}{12}\right)^{12 \cdot 5} \\ &= 3000 (1.01)^{60} \\ &= 5450.090096 \end{aligned}$$

\$5450.09 due if compounded monthly

Finally, if the money is compounded daily, then  $n = 365$  since there are 365 days in one year. Hence,

$$\begin{aligned} A(t) &= P \left(1 + \frac{r}{n}\right)^{nt} \\ A(5) &= 3000 \left(1 + \frac{.12}{365}\right)^{365 \cdot 5} \\ &= 3000 (1.000328767)^{1825} \\ &= 5465.817399 \end{aligned}$$

\$5465.82 due if compounded daily

4. **If \$3000 is borrowed at a rate of 12% interest per year, find the amount due at the end of 5 years if the interest is compounded continuously.**

For this problem, we use the compounded continuously formula with  $P = 3000$ ,  $r = .12$ , and  $t = 5$ . Substituting everything in, we get

$$\begin{aligned} A(t) &= Pe^{rt} \\ A(5) &= 3000e^{.12 \cdot 5} \\ &= 3000(1.8221188) \\ &= 5466.356401 \end{aligned}$$

\$5466.36 due if compounded continuously

5. **Find the amount that must be invested at  $5\frac{1}{2}\%$  today in order to have \$100,000 in 20 years if the investment is compounded continuously.**

Here, we have  $A(20) = 100,000$ ,  $r = .055$ , and  $t = 20$ . Substituting into the compounded continuously interest formula, we get

$$\begin{aligned} A(t) &= Pe^{rt} \\ 100,000 &= Pe^{.055 \cdot 20} \\ 100,000 &= Pe^{1.1} \\ \frac{100,000}{e^{1.1}} &= P \\ 33287.10837 &= P \end{aligned}$$

\$33,287.11 must be invested now

6. The population of a certain city has a relative growth rate of 9% per year. The population in 1978 was 24,000. Find the projected population of the city for the year 2010.

Since 1978 is our starting date, 2010 refers to  $t = 22$ . Also, we know that  $P_0 = 24,000$  and  $r = .09$ . Substituting into our exponential growth formula, we get

$$\begin{aligned} P(t) &= P_0e^{rt} \\ P(22) &= 24,000e^{.09 \cdot 22} \\ &= 24,000e^{1.98} \\ &= 173825.8316 \end{aligned}$$

173,825 people in 2010

7. The relative growth rate for a certain bacteria population is 75% per hour. A small culture is formed and 4 hours later a count shows approximately 32,500 bacteria in a culture. Find the initial number of bacteria in the culture and estimate the number of bacteria 6 hours from the time the culture was started.

To solve for the initial number of bacteria in the culture, we will use the exponential growth formula with  $r = .75$ ,  $t = 4$ , and  $P(4) = 32,500$ . Therefore,

$$\begin{aligned} P(t) &= P_0e^{rt} \\ 32,500 &= P_0e^{.75 \cdot 4} \\ 32,500 &= P_0e^3 \\ \frac{32,500}{e^3} &= P_0 \\ 1618.079722 &= P_0 \end{aligned}$$

1618 bacteria initially

Since we have the initial population of 1618, substituting this into our exponential growth formula with  $t = 6$ , we get

$$\begin{aligned} P(t) &= P_0e^{rt} \\ P(6) &= 1618e^{.75 \cdot 6} \\ &= 1618e^{4.5} \\ &= 145647.7184 \end{aligned}$$

145,647 bacteria after 6 hours