DISCOVERY

What happens to the Law of Cosines

$$c^2 = a^2 + b^2 - 2ab\cos C$$

if $C = 90°$? What familiar theorem do you obtain?

The Law of Cosines

If A, B, and C are the measures of the angles of a triangle, and a, b, and c are the lengths of the sides opposite these angles, then

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C.$$

The square of a side of a triangle equals the sum of the squares of the other two sides minus twice their product times the cosine of their included angle.

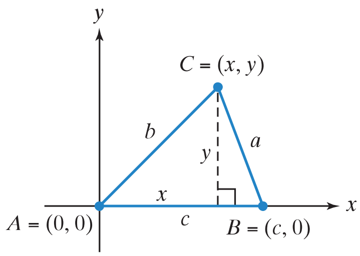

FIGURE 6.14

To prove the Law of Cosines, we place triangle ABC in a rectangular coordinate system. **Figure 6.14** shows a triangle with three acute angles. The vertex A is at the origin and side c lies along the positive x-axis. The coordinates of C are (x, y). Using the right triangle that contains angle A, we apply the definitions of the cosine and the sine.

$$\cos A = \frac{x}{b} \qquad \sin A = \frac{y}{b}$$

$$x = b\cos A \qquad y = b\sin A$$ Multiply both sides of each equation by b and solve for x and y, respectively.

Thus, the coordinates of C are $(x, y) = (b\cos A, b\sin A)$. Although triangle ABC in **Figure 6.14** shows angle A as an acute angle, if A were obtuse, the coordinates of C would still be $(b\cos A, b\sin A)$. This means that our proof applies to both kinds of oblique triangles.

We now apply the distance formula to the side of the triangle with length a. Notice that a is the distance from (x, y) to $(c, 0)$.

$a = \sqrt{(x-c)^2 + (y-0)^2}$	Use the distance formula.
$a^2 = (x-c)^2 + y^2$	Square both sides of the equation.
$a^2 = (b\cos A - c)^2 + (b\sin A)^2$	$x = b\cos A$ and $y = b\sin A$.
$a^2 = b^2\cos^2 A - 2bc\cos A + c^2 + b^2\sin^2 A$	Square the two expressions.
$a^2 = b^2\sin^2 A + b^2\cos^2 A + c^2 - 2bc\cos A$	Rearrange terms.
$a^2 = b^2(\sin^2 A + \cos^2 A) + c^2 - 2bc\cos A$	Factor b^2 from the first two terms.
$a^2 = b^2 + c^2 - 2bc\cos A$	$\sin^2 A + \cos^2 A = 1$

The resulting equation is one of the three formulas for the Law of Cosines. The other two formulas are derived in a similar manner.

1 Use the Law of Cosines to solve oblique triangles.

Solving Oblique Triangles

If you are given two sides and an included angle (SAS) of an oblique triangle, none of the three ratios in the Law of Sines is known. This means that we do not begin solving the triangle using the Law of Sines. Instead, we apply the Law of Cosines and the following procedure:

Solving an SAS Triangle

1. Use the Law of Cosines to find the side opposite the given angle.
2. Use the Law of Sines to find the angle opposite the shorter of the two given sides. This angle is always acute.
3. Find the third angle by subtracting the measure of the given angle and the angle found in step 2 from 180°.

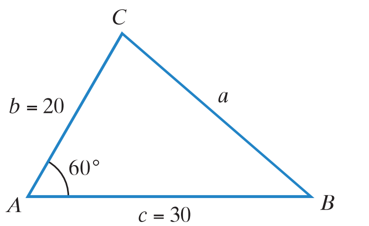

FIGURE 6.15 Solving an SAS triangle

EXAMPLE 1 Solving an SAS Triangle

Solve the triangle in **Figure 6.15** with $A = 60°$, $b = 20$, and $c = 30$. Round lengths of sides to the nearest tenth and angle measures to the nearest degree.

SOLUTION

We are given two sides and an included angle. Therefore, we apply the three-step procedure for solving an SAS triangle.

Step 1 Use the Law of Cosines to find the side opposite the given angle. Thus, we will find a.

$$a^2 = b^2 + c^2 - 2bc\cos A$$ Apply the Law of Cosines to find a.

$$a^2 = 20^2 + 30^2 - 2(20)(30)\cos 60°$$ $b = 20$, $c = 30$, and $A = 60°$.

$$= 400 + 900 - 1200(0.5)$$ Perform the indicated operations.

$$= 700$$

$$a = \sqrt{700} \approx 26.5$$ Take the square root of both sides and solve for a.

In this example, we know the exact value of $\cos 60°$: $\cos 60° = 0.5$. If the exact value of the cosine is not available, you can calculate $b^2 + c^2 - 2bc\cos A$ in one step with a calculator.

Step 2 Use the Law of Sines to find the angle opposite the shorter of the two given sides. This angle is always acute. The shorter of the two given sides is $b = 20$. Thus, we will find acute angle B.

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$ Apply the Law of Sines.

$$\frac{20}{\sin B} = \frac{\sqrt{700}}{\sin 60°}$$ We are given $b = 20$ and $A = 60°$. Use the value of a, $\sqrt{700}$, from step 1.

$$\sqrt{700}\sin B = 20\sin 60°$$ Cross multiply: If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$.

$$\sin B = \frac{20\sin 60°}{\sqrt{700}} \approx 0.6547$$ Divide by $\sqrt{700}$ and solve for $\sin B$.

$$B \approx 41°$$ Find $\sin^{-1} 0.6547$ using a calculator.

Step 3 Find the third angle. Subtract the measure of the given angle and the angle found in step 2 from 180°.

$$C = 180° - A - B \approx 180° - 60° - 41° = 79°$$

The solution is $a \approx 26.5$, $B \approx 41°$, and $C \approx 79°$.

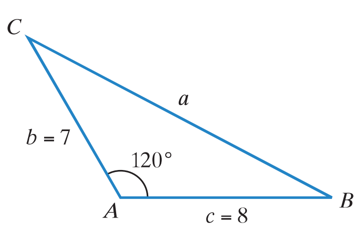

FIGURE 6.16

Check Point 1 Solve the triangle shown in **Figure 6.16** with $A = 120°$, $b = 7$, and $c = 8$. Round as in Example 1.

If you are given three sides of a triangle (SSS), solving the triangle involves finding the three angles. We use the following procedure:

Solving an SSS Triangle

1. Use the Law of Cosines to find the angle opposite the longest side.
2. Use the Law of Sines to find either of the two remaining acute angles.
3. Find the third angle by subtracting the measures of the angles found in steps 1 and 2 from 180°.

EXAMPLE 2 Solving an SSS Triangle

Solve triangle ABC if $a = 6, b = 9$, and $c = 4$. Round angle measures to the nearest degree.

SOLUTION

We are given three sides. Therefore, we apply the three-step procedure for solving an SSS triangle. The triangle is shown in **Figure 6.17**.

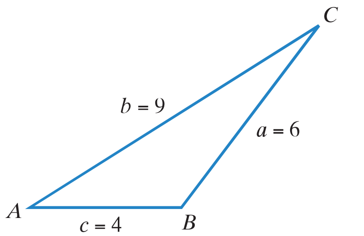

FIGURE 6.17 Solving an SSS triangle

Step 1 Use the Law of Cosines to find the angle opposite the longest side. The longest side is $b = 9$. Thus, we will find angle B.

$$b^2 = a^2 + c^2 - 2ac\cos B \quad \text{Apply the Law of Cosines to find } B.$$

$$2ac\cos B = a^2 + c^2 - b^2 \quad \text{Solve for } \cos B.$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos B = \frac{6^2 + 4^2 - 9^2}{2 \cdot 6 \cdot 4} = -\frac{29}{48} \quad a = 6,\ b = 9, \text{ and } c = 4.$$

Using a calculator, $\cos^{-1}\left(\frac{29}{48}\right) \approx 53°$. Because $\cos B$ is negative, B is an obtuse angle. Thus,

$$B \approx 180° - 53° = 127°.$$

Because the domain of $y = \cos^{-1} x$ is $[0, \pi]$, you can use a calculator to find $\cos^{-1}\left(-\frac{29}{48}\right) \approx 127°$.

Step 2 Use the Law of Sines to find either of the two remaining acute angles. We will find angle A.

$$\frac{a}{\sin A} = \frac{b}{\sin B} \quad \text{Apply the Law of Sines.}$$

$$\frac{6}{\sin A} = \frac{9}{\sin 127°} \quad \text{We are given } a = 6 \text{ and } b = 9. \text{ We found that } B \approx 127°.$$

$$9\sin A = 6\sin 127° \quad \text{Cross multiply.}$$

$$\sin A = \frac{6\sin 127°}{9} \approx 0.5324 \quad \text{Divide by 9 and solve for } \sin A.$$

$$A \approx 32° \quad \text{Find } \sin^{-1} 0.5324 \text{ using a calculator.}$$

Step 3 Find the third angle. Subtract the measures of the angles found in steps 1 and 2 from 180°.

$$C = 180° - B - A \approx 180° - 127° - 32° = 21°$$

The solution is $B \approx 127°$, $A \approx 32°$, and $C \approx 21°$. • • •

GREAT QUESTION!

In Step 2, do I have to use the Law of Sines to find either of the remaining angles?

No. You can also use the Law of Cosines to find either angle. However, it is simpler to use the Law of Sines. Because the largest angle has been found, the remaining angles must be acute. Thus, there is no need to be concerned about two possible triangles or an ambiguous case.

Check Point 2 Solve triangle ABC if $a = 8, b = 10$, and $c = 5$. Round angle measures to the nearest degree.

2 Solve applied problems using the Law of Cosines.

Applications of the Law of Cosines

Applied problems involving SAS and SSS triangles can be solved using the Law of Cosines.

EXAMPLE 3 An Application of the Law of Cosines

Two airplanes leave an airport at the same time on different runways. One flies on a bearing of N66°W at 325 miles per hour. The other airplane flies on a bearing of S26°W at 300 miles per hour. How far apart will the airplanes be after two hours?

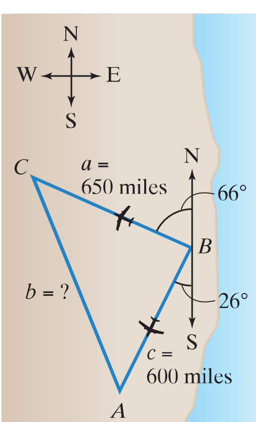

FIGURE 6.18

SOLUTION

After two hours, the plane flying at 325 miles per hour travels $325 \cdot 2$ miles, or 650 miles. Similarly, the plane flying at 300 miles per hour travels 600 miles. The situation is illustrated in **Figure 6.18**.