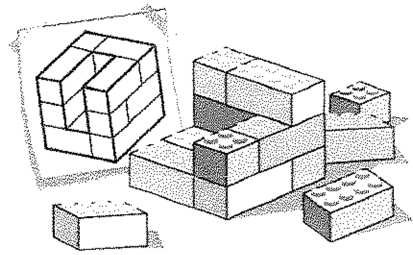
Axiomatic Systems

An **axiomatic system** is a set of statements, some of which, **axioms** or **postulates**, we accept without proof. The other statements in an axiomatic system include theorems, which are truths that can be derived from the axioms. Mathematicians accept undefined terms and definitions as true so that they can build a consistent system.



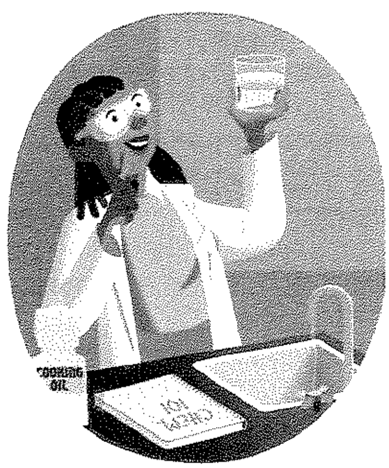
Like the pieces of a model which rest securely on others, the theorems of an axiomatic system rest on axioms and other theorems. If not placed with perfect logic, the whole system can come tumbling down.

Geometry, like all axiomatic systems, is tied together by logic. This logic is usually expressed in a convincing argument or **proof**. You can also use a proof to show that something is *not* true.

Inductive Reasoning

Have you ever done an experiment for a science project, using the scientific method to investigate a question? If so, then you have used inductive reasoning.

Consider a young scientist investigating oil and water. She mixes different amounts of oil and water and observes that the two liquids separate. She makes the following hypothesis: oil and water don't mix. The scientist is using inductive reasoning since she is making a generalization based on a pattern of observations.



Inductive reasoning is the process of observing and recording data, looking for patterns, and making generalizations from the observations.

MORE HELP
See 007, 058, 061, 141, 157, 164

Inductive reasoning is the basis of the scientific method. It is also important in mathematics. In science, you can use inductive reasoning to make and test a hypothesis. In mathematics, you can use inductive reasoning to investigate and make a conjecture. A **conjecture** is an unproven statement that is based on observations. It is like a hypothesis in the scientific method. Using the scientific method, you test the hypothesis. In mathematics, you try to prove the conjecture.

MORE ▸

Reasoning in geometry often consists of the following three steps.

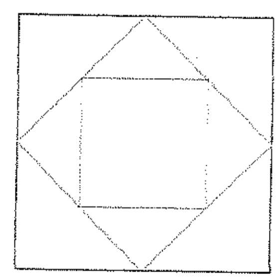
Step 1 Inductive Reasoning: Look for a pattern. Look at examples and try to discover a pattern. You can use tables and diagrams to help investigate the examples.

Step 2 Inductive Reasoning: Make a conjecture. Use the examples to make a generalization. This generalization is the conjecture. It is unproven and based upon your observations. If possible, discuss the conjecture with others and modify it if needed.

Step 3 Deductive Reasoning: Verify the conjecture. Use logical reasoning to try to show that the conjecture is true for all cases, not just for the cases that you have observed.

EXAMPLE 1: Use reasoning to find what figure is formed when the midpoints of the sides of a square are connected in order.

Step 1: Look for a pattern by investigating several cases.



The blue figure appears to be a square and the red figure does, as well.

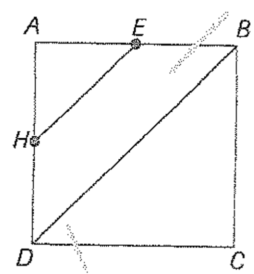
Step 2: Make a conjecture.

The figure formed when the midpoints of the sides of a square are connected in order is another square.

Step 3: Verify the conjecture.

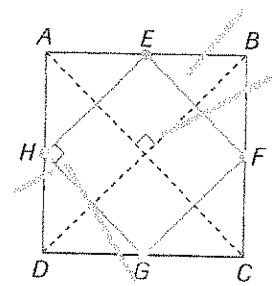
1. If you draw diagonal $\overline{DB}$ of the given square $ABCD$, $\triangle ABD$ is formed.

2. By the Midsegment Theorem, the segment connecting the midpoints of two sides of a triangle is parallel to the third side and half its length. So, $\overline{HE}$ is parallel to $\overline{DB}$ and $HE = \frac{1}{2} DB$.



3. $\overline{HG}$ is parallel to $\overline{AC}$ by the Midsegment Theorem.

4. Since the diagonals of a square are congruent, each side of the figure formed is equal to half the measure of the diagonal. So, $\overline{EF}$, $\overline{FG}$, $\overline{GH}$, and $\overline{HE}$ are all congruent, making the figure either a rhombus or a square.



5. $\overline{DB}$ is perpendicular to $\overline{AC}$ because the diagonals of a square are perpendicular.

6. $\overline{HE}$ is perpendicular to $\overline{HG}$ since each is parallel to one of a pair of perpendicular segments.

Each pair of congruent adjacent sides of the quadrilateral forms an angle that measures 90°. Therefore, the figure formed when the midpoints of the sides of a square are connected in order is another square.

To prove that a conjecture is true, you must prove that it is true in all cases. However, to prove that a conjecture is false, you only need to find one case where it is not true. This case is a **counterexample**.

Not all conjectures can be shown to be true or false. These conjectures are called unproven or undecided. A famous unproven conjecture is the Goldbach Conjecture: *Every even number greater than two can be written as the sum of two primes.* For example, 12 = 5 + 7 and 24 = 17 + 7.

MATH ALERT Don't Leap to Conclusions About the Truth of a Conjecture

Remember, although something is true for several, or even many, cases, it may not be true for *all* cases.

EXAMPLE 1: Find a counterexample for the following conjecture: $n < n^2$.

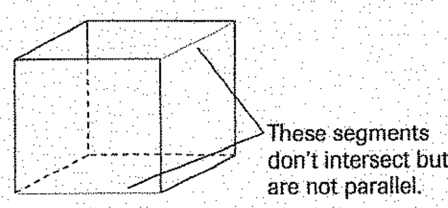
n	n^2
10	100
7	49
3	9
2.9	8.41
1.1	1.21

You might jump to the conclusion that $n > n^2$. However, this is *not* true.
Notice that in the table, n and n^2 are greater than 1. Choose a value for n that is less than 1.

Counterexample: If n is 0.8, n^2 is 0.64. 0.8 > 0.64.

★ If the length of a side of a square is 0.8 units, then the area is 0.64 square units. This is a counterexample to the conjecture that $n < n^2$.

EXAMPLE 2: You may think that any pair of lines that doesn't intersect must be parallel, but think about the edges of a box—some don't intersect and are also not parallel.



007

Deductive Reasoning

You use deductive reasoning when you verify a conjecture.

More Help
See 005, 009

This is the type of reasoning that doctors often use to make a diagnosis. For example, a doctor might compare a list of your symptoms with the symptoms associated with a specific illness. The doctor may then draw blood and have it tested for antibodies that correspond to this illness. Based on the evidence and certain conditions being present, the doctor deduces that a given illness is present.

Deductive reasoning uses facts, definitions, and accepted properties in a logical order to present a convincing argument. A **logical argument** consists of a conjecture (or set of conjectures) and a conclusion. **Logicians** are mathematicians that specialize in logical arguments. They use symbols to represent statements and logical relationships. Look at 005 to see an example of the use of deductive reasoning to verify a conjecture.

008

Logical Statements

Suppose you are asked to draw a figure that is a triangle. Would you always draw a right triangle? Probably not. However, if you were asked to draw a triangle with one right angle, you would draw a right triangle. This is the idea behind necessary and sufficient conditions in mathematical statements.

More Help
See 009–011

A **necessary condition** is required to be true in order for something else to be true. For example, it is necessary for at least three pairs of corresponding parts to be congruent to prove two triangles are congruent. If the necessary condition is false, then what it is a condition for is also false. If you don't have at least three pairs of congruent corresponding parts, then you can't possibly have congruent triangles. However, this condition may not be enough to assure that you have congruent triangles.

A **sufficient condition** is required to be true based upon the truth of a statement. For example, three pairs of congruent parts that assure congruent triangles are side-side-side, side-angle-side, or angle-side-angle.

The table below shows how basic statements are related to necessary and sufficient conditions:

Basic Statement	Conditions	Examples
If p then q $p \rightarrow q$	p only if q p is a sufficient condition for q	p = figure is a square q = figure is a quadrilateral If a figure is a square, then it is a quadrilateral.
If q then p $q \rightarrow p$	q only if p p is a necessary condition for q	p = figure has two acute angles q = figure is a $\triangle$ with obtuse $\angle$ If a figure is a triangle with an obtuse angle, then it has two acute angles.
p if and only if q $p \leftrightarrow q$	p is a necessary and sufficient condition for q	p = equilateral $\triangle$ q = equiangular $\triangle$ A triangle is equilateral if and only if it is equiangular.

More Help See 009

More Help See 009–010

More Help See 011

Conditional Statements

009

You have probably recognized how important it is to have precise meanings for words and figures in geometry. It is also very important to know the mathematical meanings of small words such as *and*, *if*, *then*, and *or*, and to understand the meanings of symbols such as >, =, and <.

More Help See 007

A statement in the form *If . . ., then . . .* is called a **conditional statement**. A conditional statement has two parts, the hypothesis and the conclusion. The *if* part of the statement is the **hypothesis**. It is denoted by p. The *then* part of the statement is the **conclusion**. It is denoted by q.

When you translate conditional statements into symbolic form, you use letters such as p, q, r, *and* s to stand for simple statements. The statements are either true or false. Logical statements can be written symbolically.

Write	Say
$p \rightarrow q$	*if p then q* or *p implies q*
$p \leftrightarrow q$	*p if and only if q*
$\sim p$	*not p*

More ▸

EXAMPLE: Write the statement, *If today is Saturday, then I don't have school*, in symbolic form.

p: *today is Saturday*

q: *I don't have school.*

★ $p \rightarrow q$: *If today is Saturday, then I don't have school*

Here is a summary of the different types of conditional statements.

Basic Statement	Symbolic Form	Example
Conditional	$p \rightarrow q$	**If** both pairs of opposite sides of a quadrilateral are congruent, **then** it is a parallelogram.
Converse	$q \rightarrow p$	**If** a quadrilateral is a parallelogram, **then** both pairs of opposite sides are congruent.
Negation	$\sim p$	Both pairs of opposite sides of a quadrilateral are **not** congruent.
Inverse	$\sim p \rightarrow \sim q$	**If** both pairs of opposite sides of a quadrilateral are **not** congruent, **then** it is **not** a parallelogram.
Contrapositive	$\sim q \rightarrow \sim p$	**If** a quadrilateral is **not** a parallelogram, **then** both pairs of opposite sides are **not** congruent.

MORE HELP See 010

MORE HELP See 012

MORE HELP See 013

MORE HELP See 014

010

Converse Statements

The **converse** of a conditional statement is formed by reversing the hypothesis and the conclusion, so the converse of the statement *if p then q* is *if q then p*. Or symbolically, if the statement is $p \rightarrow q$ then the converse is $q \rightarrow p$. The converse of a statement may or may not be true.

EXAMPLE 1: Write the converse of the statement, *If an animal is a collie, then it is a dog.* Is the converse of this statement true?

Statement: If an animal is a collie, then it is a dog.

Reverse the hypothesis and the conclusion.

Converse: If an animal is a dog, then it is a collie.

You know, however, that while a dog could be a collie, it could instead be a poodle, or a hound, or some other breed.

★ The converse is: *If an animal is a dog, then it is a collie.* Since a dog may or may not be a collie, the converse is false.

EXAMPLE 2: Is the converse of this statement true?

MORE HELP See 072

Statement: If two angles are vertical angles, then the angles are congruent.

Converse: If two angles are congruent, then they are vertical angles.

You know that two angles can be congruent without being vertical angles.

★ The statement is true but the converse is false. Congruent angles do not have to be vertical angles.

EXAMPLE 3: Is the converse of this statement true?

Statement: If two nonvertical lines are parallel, then they have the same slope.

Converse: If two nonvertical lines have the same slope, then they are parallel.

The converse is a true statement.

★ Both the statement and its converse are true.

MORE HELP See 107

When a statement and its converse are both true, as in Example 3, they can be combined into a biconditional statement: *Two nonvertical parallel lines are parallel if and only if they have the same slope.*

If and only if is sometimes abbreviated as *iff*.

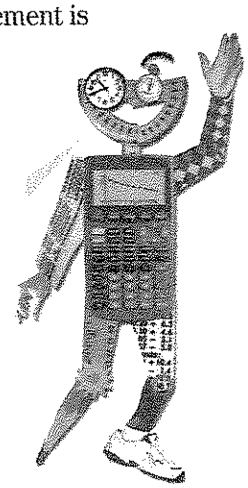
Biconditional Statements

011

When a statement and its converse are both true, we sometimes write them as a **biconditional statement**, *p if and only if q*. The statement is equivalent to $p \rightarrow q$ *and* $q \rightarrow p$.

Write: $p \leftrightarrow q$
Say: *p if and only if q*

A good definition is biconditional. For example, congruent segments have equal measure. Although *if and only if* is not included in the definition, it is implied. This means that equivalent forms of the definition are: *If two segments are congruent, then they have equal measure and if two segments have equal measure, then they are congruent.*



012

Negations

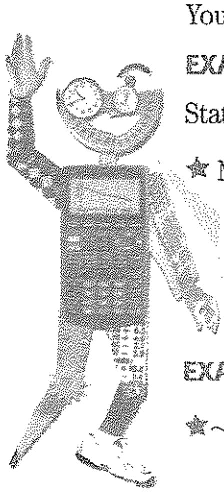
Have you ever been joking with your friends and said something like, *I like sardines. NOT!* Such a statement is called a negation.

The **negation** of a statement p, called not-p, is a statement that is true when p is false and is false when p is true.

Write: $\sim p$

Say: *not p*

You can often write a negation by inserting the word *not* into the statement.



EXAMPLE 1: Write the negation of the statement.

Statement: *Natalie is training for the race.*

★ Negation: *Natalie is not training for the race.*

If the statement is already negative, then the negation takes out the not in the statement.

EXAMPLE 2: Write the negation for p: The polygon is not a hexagon.

★ $\sim p$: The polygon is a hexagon.

013

Inverse Statements

The **inverse** of a statement is formed by negating both the hypothesis and the conclusion. The inverse of the statement, *if p then q* is *if not p then not q*. Symbolically, write the inverse of $p \rightarrow q$ as $\sim p \rightarrow \sim q$. The inverse of a statement may or may not be true.

EXAMPLE 1: Write the inverse of the statement.

Statement: *If two lines are parallel, then the distance between them is constant.*

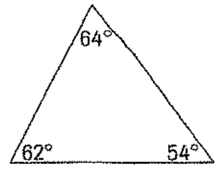
★ Inverse: *If two lines are not parallel, then the distance between them is not constant.*

EXAMPLE 2: Write the inverse of this statement. Determine whether it is true.

Statement: *If a triangle is equilateral, then it has three acute angles.*

Inverse: *If a triangle is not equilateral, then it does not have three acute angles.*

Remember, you only need one counterexample to show that a statement is not true. Look at the diagram. The triangle is not equilateral, but it does have three acute angles.



★ The statement is true, but the inverse is false.

EXAMPLE 3: Write the inverse of this statement. Determine whether it is true.

Statement: *If two angles are supplementary*, ***then*** *the sum of their measures is 180°.*

Inverse: *If two angles are not supplementary*, ***then*** *the sum of their measures is not 180°.*

By definition, for two angles to be supplementary, they must have a sum of 180°. If two angles are not supplementary, then their sum cannot be 180°.

★ Both the statement and its inverse are true.

Contrapositive Statements

The **contrapositive** of a conditional statement is formed when both the hypothesis and the conclusion are reversed *and* negated. As a result, the contrapositive of the statement, *if p then q* is *if not q then not p*.

Write: $\sim q \rightarrow \sim p$

Say: *if not q, then not p*

EXAMPLE: Write the contrapositive of this statement. Decide whether it is true.

Statement: *If a figure is a square*, ***then*** *it is a parallelogram.*

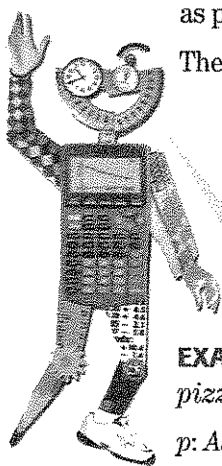
★ Contrapositive: *If a figure is not a parallelogram*, then *it is not a square.* The contrapositive is true.

Notice that in the example, the statement and its contrapositive are both true. It turns out that this is always the case. The **law of the contrapositive** says that a statement and its contrapositive are logically equivalent. You may replace one with the other.

016

Law of Detachment and Law of Syllogism

Two patterns of logical reasoning that use conditional statements are the law of detachment and the law of syllogism. You might think of both laws as plain common sense and, in a way, that's what logic is.



The **law of detachment** says:

if $p \rightarrow q$ is true
and p is true
then q is true

The law of detachment is sometimes referred to by its Latin name, *Modus Ponens*.

EXAMPLE 1: Assume that, *if Allie goes to the mall, then she will have pizza* is true. Today, Allie goes to the mall. Will Allie have pizza? Why?

p: *Allie goes to the mall*
q: *she will have pizza.*
$p \rightarrow q$: *If Allie goes to the mall, then she will have pizza.*

Since $p \rightarrow q$ is assumed to be true and p is also true, the law of detachment says that q, *Allie will have pizza*, will occur.

★ Allie will have pizza.

The **law of syllogism** says:

if $p \rightarrow q$ is true
and $q \rightarrow r$ is true
then $p \rightarrow r$ is true

The law of syllogism is sometimes called the Chain Rule.

EXAMPLE 2: If Clayton gets a job, then he will earn money. If Clayton earns money, then he will buy a mountain bike. If both statements are true, what can you conclude?

p: *Clayton gets a job*
q: *he will earn money*
r: *he will buy a mountain bike*

$p \rightarrow q$: *if Clayton gets a job, then he will earn money*
$q \rightarrow r$: *if Clayton earns money, then he will buy a mountain bike*
$p \rightarrow r$: *if Clayton gets a job, then he will buy a mountain bike*

Since $p \rightarrow q$ and $q \rightarrow r$ are both true, then $p \rightarrow r$ must be true by the law of syllogism.

★ You can conclude: *if Clayton gets a job, then he will buy a mountain bike.*