**CALCULUS IS BUILT ON THE NOTION OF LIMIT.**

WE LIVE IN A WORLD OF LIMITS. AND THIS IS WHERE CALCULUS COMES IN. MOST OF THE LIMITS THAT INTEREST US CAN BE VIEWED AS NUMERICAL LIMITS TO VALUES OF FUNCTIONS, AND CALCULUS IS THE RIGHT MATHEMATICS FOR FINDING THE LIMITING VALUES OF FUNCTIONS.

# 2.1 LIMITS OF FUNCTION VALUES - DAY 1

Objectives:

- Estimate a limit using a numerical or graphical approach
- Know different ways a limit can fail to exist

## An Introduction to Limits

- If the inputs of a function get closer and closer to some specific value $c$, will the outputs get closer and closer to some specific value $L$?

Evaluate the function $f(x) = x/\left(\sqrt{x+1} - 1\right)$ at several points near $x = 0$ and use the results to estimate the limit

$$\lim_{x\to 0} \frac{x}{\sqrt{x+1} - 1}.$$

**Solution** The table lists the values of $f(x)$ for several $x$-values near 0.

$x$ approaches 0 from the left. → ← $x$ approaches 0 from the right.

| $x$ | −0.01 | −0.001 | −0.0001 | 0 | 0.0001 | 0.001 | 0.01 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 1.99499 | 1.99950 | 1.99995 | ? | 2.00005 | 2.00050 | 2.00499 |

$f(x)$ approaches 2. → ← $f(x)$ approaches 2.

From the results shown, what can we estimate the limit to be?

Example 1: Estimating a Limit Numerically

Estimate the following limit numerically by completing the table.

$$\lim_{x\to 2} \frac{x^2 - 3x + 2}{x - 2}$$

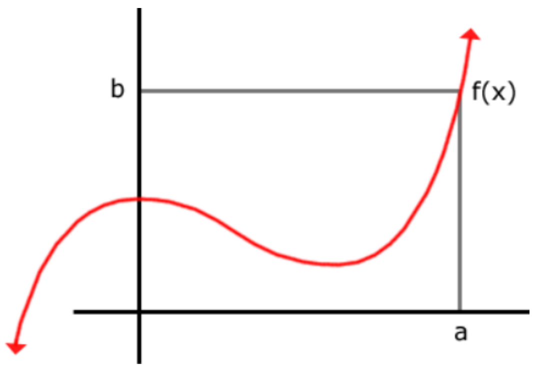
| $x$ | 1.75 | 1.9 | 1.99 | 1.999 | 2 | 2.001 | 2.01 | 2.1 | 2.25 |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | | | | | | | | | |

## Observation:

- In Example 1, the function is undefined at $x = 2$ and yet $f(x)$ appears to be approaching a limit as $x$ approaches 0.
  - Important!

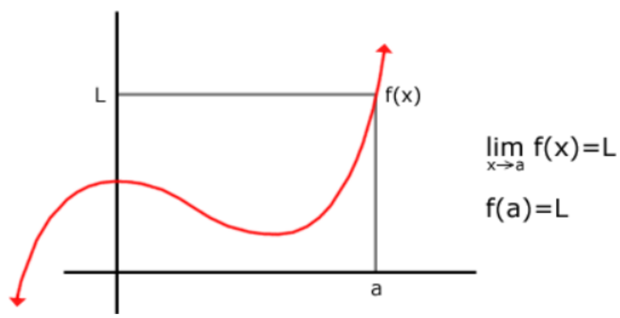
    The existence or nonexistence of ______________________________ has no bearing on the existence of the limit of $f(x)$ as $x$ approaches $c$.

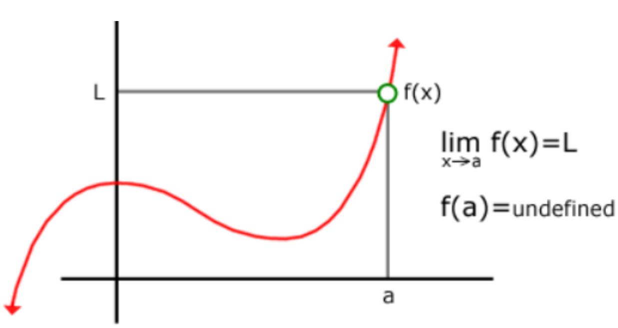
## A Graphical Approach

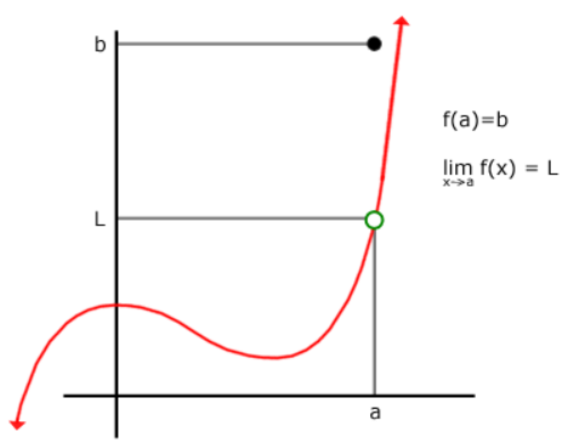



The limit of f(x) as x ⟶ a is b.

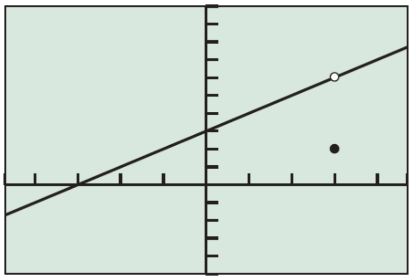
- Three Types of Limits
  - The limit and the function each have the ____________________
  - The function has a limit at a, but the function is ____________________ at a
  - The function may have a limit of one value but a function value of ________________________________



Example 2: A Graphical Approach



What happens to the values of the function

$$f(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3} & \text{if } x \neq 3 \\ 2 & \text{if } x = 3 \end{cases}$$

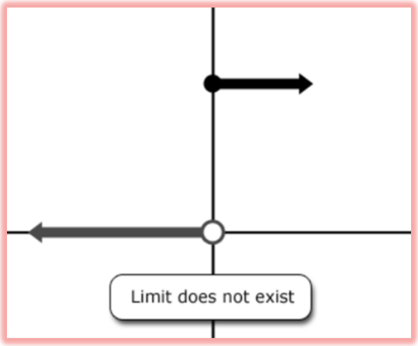
as $x$ approaches 2?

> The limit of a function $f(x)$ as $x$ approaches $c$ never depends on what happens when $x = c$. The limit, if it exists at all, is determined solely by the values $f$ has when $x \neq c$. That is, when we find limits, we look at values near $c$ but not equal to $c$.

- An Informal Definition of Limits
  - If $f(x)$ becomes arbitrarily close to a single number $L$ as $x$ approaches $c$ from either side, the limit of $f(x)$, as $x$ approaches $c$, is $L$.

______________________

- Limits That Fail to Exist




  - Behaviors That Differ from the Right & Left
    - $f(x)$ approaches a different value from the right side of $c$ than it approaches from the left side
  - Left- & Right-Hand Limits
    - Left-hand: the limit of $f$ as $x$ approaches $c$ from the left

______________________

    - Right-hand: the limit of $f$ as $x$ approaches $c$ from the right

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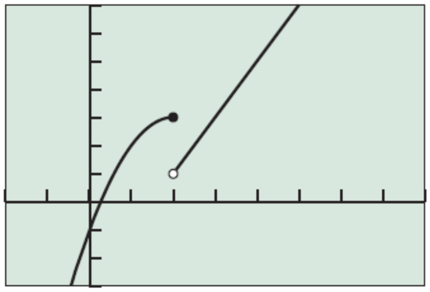
Example 3: Finding Left- & Right-Hand Limits

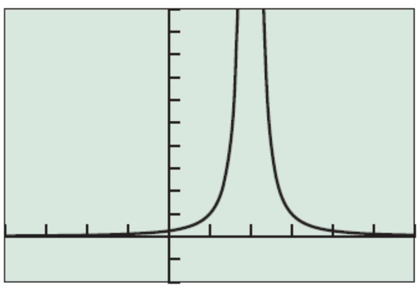


Find $\lim_{x \to 2^-} f(x)$ & $\lim_{x \to 2^+} f(x)$ where

$$f(x) = \begin{cases} -x^2 + 4x - 1 & \text{if } x \le 2 \\ 2x - 3 & \text{if } x > 2 \end{cases}$$

- Limits That Fail to Exist (continued)
  - Unbounded Behavior
    - $f(x)$ ________________________________________ as $x$ approaches $c$

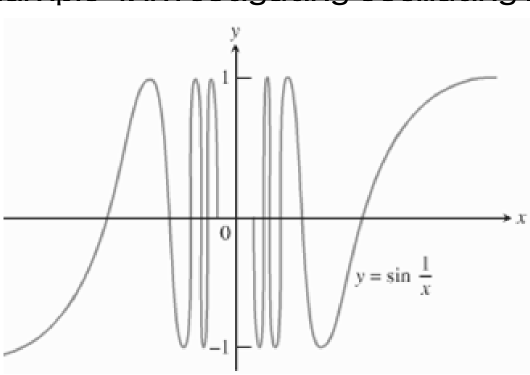
Example 4: Investigating Unbounded Limits



Find $\lim_{x \to 2} \frac{1}{(x-2)^2}$.

- Limits That Fail to Exist (continued)
  - Oscillating Behavior
    - $f(x)$ oscillates ________________________________________ as $x$ approaches $c$

Example 4: Investigating Oscillating Behavior




Discuss the existence of the limit

$$\lim_{x \to 0} \sin \frac{1}{x}$$

## One-sided & Two-sided Limits

- A function $f(x)$ has a limit as $x$ approaches $c$ if and only if ______________

______________________________________________________________

$$\lim_{x\to c} f(x) = L \Leftrightarrow \lim_{x\to c^-} f(x) = L \ \& \ \lim_{x\to c^+} L$$

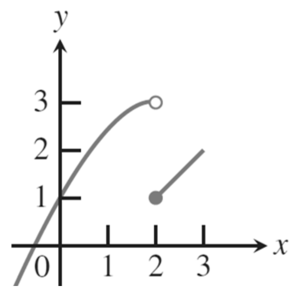
Examples: Finding One- & Two-sided Limits

Use the given graph to find the limits or to explain why the limits do not exist.

5. (a) $\lim_{x\to 2^-} f(x)$

(b) $\lim_{x\to 2^+} f(x)$

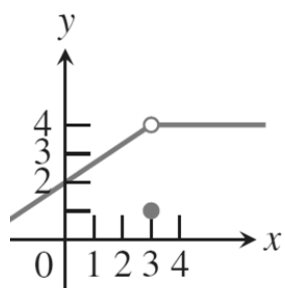
(c) $\lim_{x\to 2} f(x)$




6. (a) $\lim_{x\to 3^-} f(x)$

(b) $\lim_{x\to 3^+} f(x)$

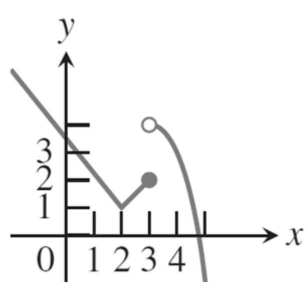
(c) $\lim_{x\to 3} f(x)$




7. (a) $\lim_{x\to 3^-} f(x)$

(b) $\lim_{x\to 3^+} f(x)$

(c) $\lim_{x\to 3} f(x)$



# 2.1 Limits of Function Values - Day 2

Objectives:

- Evaluate a limit using the properties of limits
- Evaluate a limit analytically

## General Rules for Calculating Limits

- If $f$ is the identity function $f(x) = x$, then for any value of $c$,

  ______________________________

- If $f$ is the constant function $f(x) = k$, then for any value of $c$,

  ______________________________

## Properties of Limits

If $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ both exist, then

| | |
|---|---|
| **1. Sum Rule** | $\lim_{x\to c} (f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$ |
| **2. Difference Rule** | $\lim_{x\to c} (f(x) - g(x)) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$ |
| **3. Product Rule** | $\lim_{x\to c} (f(x) \cdot g(x)) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$ |
| **4. Constant Multiple Rule** | $\lim_{x\to c} (k \cdot g(x)) = k \cdot \lim_{x\to c} g(x)$ |
| **5. Quotient Rule** | $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, provided $\lim_{x\to c} g(x) \neq 0$ |
| **6. Power Rule** | $\lim_{x\to c} (f(x))^n = (\lim_{x\to c} f(x))^n$ for $n$ a positive integer |
| **7. Root Rule** | $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)}$ for $n \geq 2$ a positive integer, provided $\sqrt[n]{\lim_{x\to c} f(x)}$ and $\lim_{x\to c} \sqrt[n]{f(x)}$ are real numbers. |

Examples: Using the Properties of Limits

Assume that $\lim_{x\to b} f(x) = 7$ & $\lim_{x\to b} g(x) = -3$. Find ...

1. $\lim_{x\to b}(f(x) + g(x))$
2. $\lim_{x\to b}(f(x) \cdot g(x))$
3. $\lim_{x\to b} 4g(x)$
4. $\lim_{x\to b} \frac{f(x)}{g(x)}$

## Strategies for Finding Limits Analytically

- Continuous Functions
  - Limits can be found via direct ___________________
- Limits of Polynomial & Rational Functions
  - If $p$ is a polynomial function and $c$ is a real number, then:

$$\lim_{x\to c} p(x) = p(c)$$

  - If $r$ is a rational function given by $r(x) = p(x)/q(x)$ and $c$ is a real number such that $q(c) \neq 0$, then:

$$\lim_{x\to c} r(x) = r(c) = \frac{p(c)}{q(c)}$$

<u>Examples: Finding Limits of Continuous Functions</u>
Find the limit by direct substitution.

5. $\lim_{x\to 2}(x^3 - 2x + 3)$

6. $\lim_{x\to -4}(x-3)^2$

7. $\lim_{x\to 7}\frac{5x}{\sqrt{x+2}}$

8. $\lim_{x\to -3}\frac{2x-3}{x+5}$

## Techniques for Rational Functions

- If substitution results with a zero in the denominator or ______________ ______________, rewrite the fraction so that the denominator does not have 0 as its limit.
  - Dividing Out Technique
    - 
  - Rationalizing Technique
    - 

Examples: Dividing Out & Rationalizing Techniques

Find the limit via dividing out:

9. $\lim_{x \to -3} \dfrac{x^2 + x - 6}{x + 3}$

Find the limit via rationalizing:

10. $\lim_{x \to 0} \dfrac{\sqrt{x+1} - 1}{x}$

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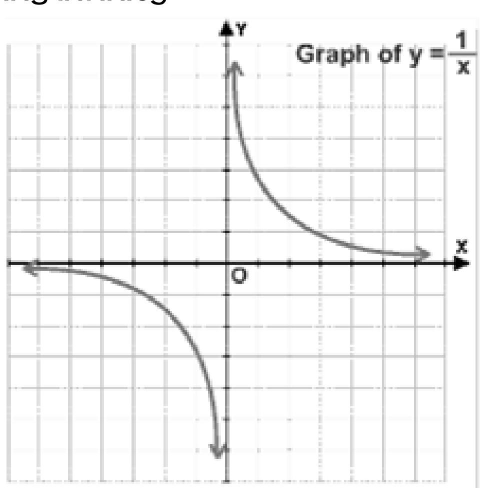
## 2.2 Limits Involving Infinity

Objectives:

- Determine infinite limits from the left and from the right
- Evaluate infinite limits of rational functions

Limits Involving Infinity




$\lim_{x \to \infty} \dfrac{1}{x} = 0^+$

$\lim_{x \to -\infty} \dfrac{1}{x} = 0^-$

$\lim_{x \to 0^+} \dfrac{1}{x} = \infty$

$\lim_{x \to 0^-} \dfrac{1}{x} = -\infty$
~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~

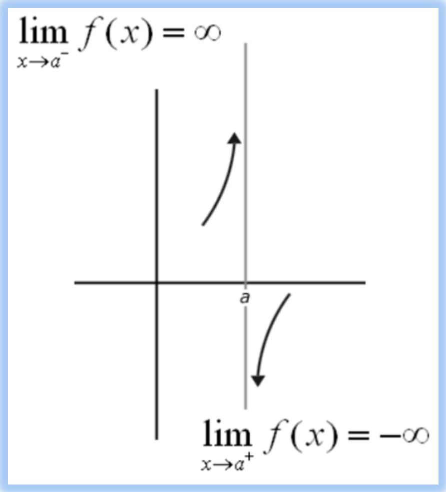
**DEFINITION Limits at Infinity**

When we write "$\lim_{x \to \infty} f(x) = L$," we mean that $f(x)$ gets arbitrarily close to $L$ as $x$ gets arbitrarily large. We say that **$f$ has a limit $L$ as $x$ approaches $\infty$**.

When we write "$\lim_{x \to -\infty} f(x) = L$," we mean that $f(x)$ gets arbitrarily close to $L$ as $-x$ gets arbitrarily large. We say that **$f$ has a limit $L$ as $x$ approaches $-\infty$**.

**INFINITE LIMITS ARE NOT LIMITS**

It is important to realize that *an infinite limit is not a limit*, despite what the name might imply. It describes a special case of a limit that does not exist. Recall that a sawhorse is not a horse and a badminton bird is not a bird.




- What is an Infinite Limit?
  - A limit in which ________________________________ as $x$ approaches $c$
    - It may be necessary to use your grapher with infinite limits, one-sided &/or two-sided limits.

<u>Examples: Determining Infinite Limits from a Graph</u>

Use a grapher to graph each function. For each function, analytically find the single real number c that is not in the domain. The graphically find the limit of $f(x)$ as $x$ approaches $c$ from the left and from the right.

| | $\lim_{x \to c^-} f(x)$ | $\lim_{x \to c^+} f(x)$ | | $\lim_{x \to c^-} f(x)$ | $\lim_{x \to c^+} f(x)$ |
|---|---|---|---|---|---|
| 1. $f(x) = \dfrac{3}{x-4}$ | ______ | ______ | 2. $f(x) = \dfrac{1}{2-x}$ | ______ | ______ |
| 3. $f(x) = \dfrac{2}{(x-3)^2}$ | ______ | ______ | 4. $f(x) = \dfrac{-3}{(x+2)^2}$ | ______ | ______ |

- Limits of Rational Functions as $x \to \pm\infty$
  - Finding Limits by Inspection
    - Compare the degree of the numerator $(N°)$ to the degree of the denominator $(D°)$

$N° < D°$ $\quad \lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = 0$

$N° = D°$ $\quad \lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \frac{a_n}{b_n}$

______________________________

$N° > D°$ $\quad \lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \pm\infty$

______________________________

Examples: Evaluating Infinite Limits of Rational Functions

5. $\lim_{x\to\infty} \frac{-15x}{7x+4}$

6. $\lim_{x\to-\infty} \frac{2x^2-3}{7x+4}$

7. $\lim_{x\to-\infty} \frac{11x+2}{2x^3-1}$

8. $\lim_{x\to-\infty} \frac{8x-3+5x^2}{2+3x^2}$

9. $\lim_{x\to\infty} \frac{-4x^3+7x}{2x^2-3x+10}$

10. $\lim_{x\to-\infty} \frac{-4x^3+7x}{2x^2-3x+10}$

## 2.3 The Sandwich Theorem & $\sin\theta / \theta$

Objective:

- Evaluate a limit using the Sandwich Theorem

- The Sandwich Theorem
  - Suppose that $g(x) \le f(x) \le h(x)$ for all $x \ne c$ in some open interval about $c$ that:

    ______________________________

    Then:

    ______________________________

## Two Special Trigonometric Limits

__________________ __________________

Examples: Applying Trigonometric Limits

1. $\lim_{x\to 0} \tan x$

2. $\lim_{x\to 0} \sec x$

## Theorem:

- If $\theta$ is measured in radians, then: __________________

Example 3: Using the Sandwich Theorem

Use the Sandwich Theorem to show that:

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0$$

Examples: Using ${\sin\theta}/{\theta}$

Find the limit.

4. $\lim_{x\to 0} \frac{\sin 7x}{7x}$

5. $\lim_{x\to 0} \frac{\sin 2x}{5x}$

6. $\lim_{x\to 0} \frac{\tan 2x}{5x}$

7. $\lim_{x\to 0} \frac{x + x\cos x}{\sin x \cos x}$

# 2.4 LIMITS & CONTINUITY

Objectives:

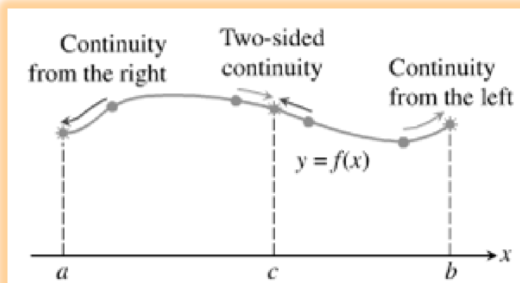
- Determine continuity at a point, on an open interval, and on a closed interval
- Determine whether a discontinuity is removable

## Continuous Functions

- A function is continuous if it is ________________ and there are no ________________ or ___________; if it is continuous at each point of its ________________

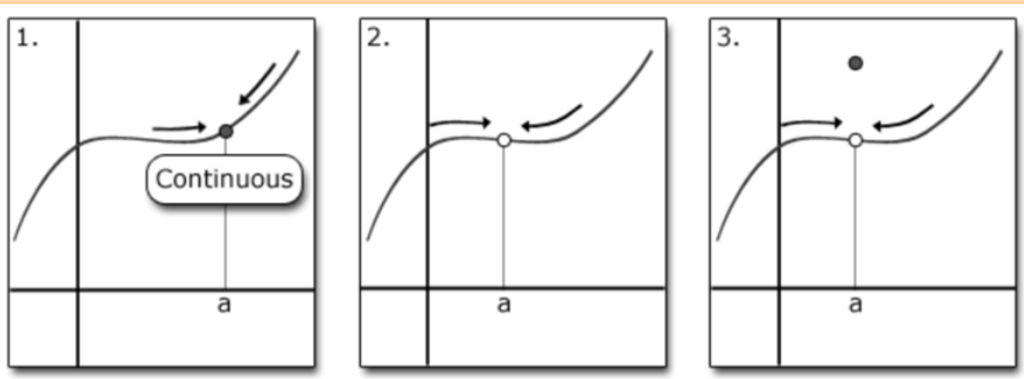
## Continuity on an Open Interval

- A function is continuous on an open interval (a, b) if it is continuous at __________________ in the interval.
  - A function that is continuous on the entire real line $(-\infty, \infty)$ is "everywhere continuous."




## Continuity @ a Point

- **The Continuity Test**
  - A function $f(x)$ is continuous at $c$ if the following three conditions are met:
    1. ____________________________________
    2. ____________________________________
    3. ____________________________________




In these three pictures, we can see three different possibilities for functions. In the first one, the function equals the limit value and this would be a continuous function. In the 2nd and 3rd examples, the limits and function values do not equal each other.

Examples: Discontinuity @ a Point

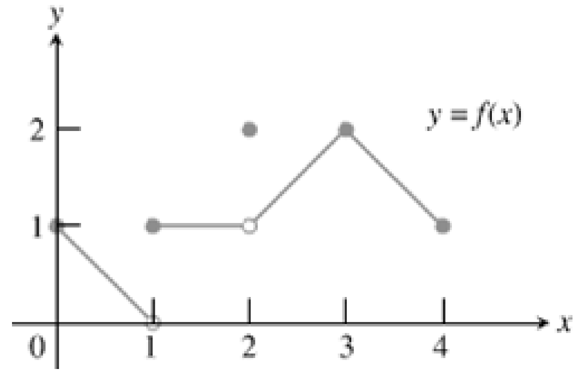
Why is $f(x)$ discontinuous at the following points?

1. $x = 1$

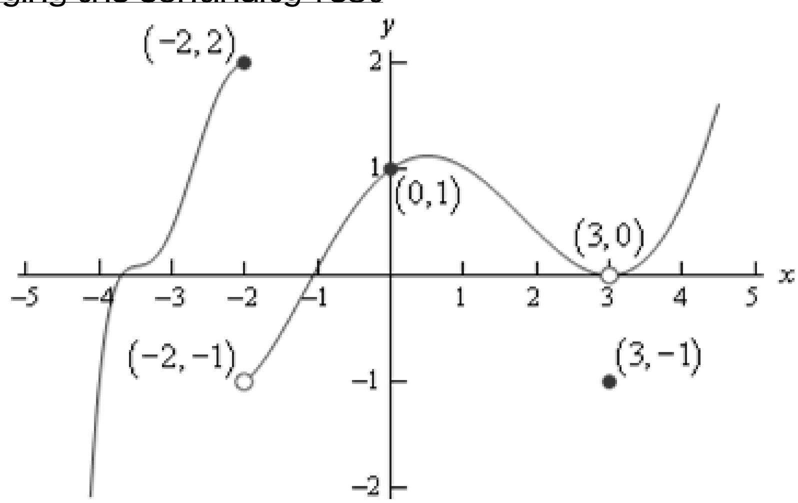
2. $x = 2$

3. Is $f(x)$ continuous at $x = 0$?
   Why or why not?

4. Is $f(x)$ continuous at $x = 4$?
   Why or why not?




Examples: Applying the Continuity Test




5. Is $f(x)$ discontinuous at $x = -2$? Why or why not?

6. Is $f(x)$ discontinuous at $x = 0$? Why or why not?

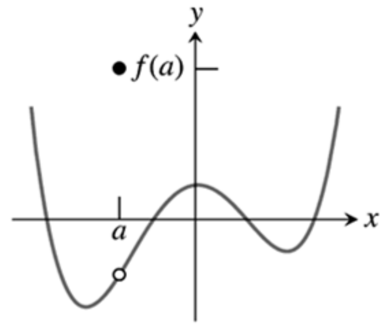
7. Is $f(x)$ discontinuous at $x = 3$? Why or why not?

## Removable Discontinuities

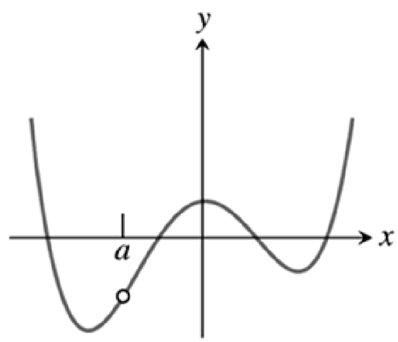
- One single type of discontinuity, called a removable discontinuity occurs whenever

  ______________________________________________

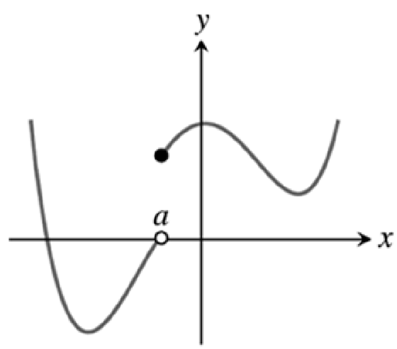
  - We remove the discontinuity by ____________ ________________________ to have the same value as ___________________
    - The ______________________________ for this to be possible.



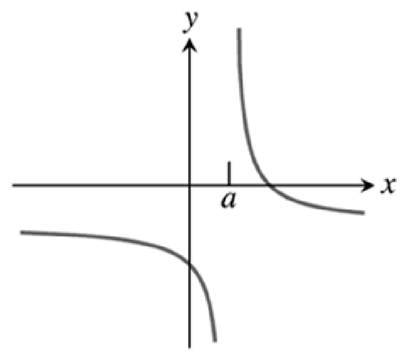
Removable discontinuity



Removable discontinuity



Jump discontinuity



Infinite discontinuity

Examples: Removing Discontinuities

8. Identify those values where $f(x)$ is discontinuous.

9. Which values, if any, can the discontinuity be removed?

10. How should that value be redefined so that $f(x)$ is continuous at $c$?