Section IV: Power, Polynomial, and Rational Functions

Module 1: Power Functions

DEFINITION: A **power function** is a function of the form $f(x) = kx^p$ where k and p are constants.

EXAMPLE: Which of the following functions are power functions? For each power function, state the value of the constants k and p in the formula $y = kx^p$.

a. $b(x) = 5(x-3)^4$ **b.** $m(x) = 7\sqrt[4]{x}$

c. $l(x) = 3 \cdot 2^x$ **d.** $s(x) = \sqrt{\frac{7}{x^5}}$

SOLUTIONS:

a. The function $b(x) = 5(x-3)^4$ is not a power function because we cannot write it in the form $y = kx^p$.

b. The function $m(x) = 7\sqrt[4]{x}$ is a power function because we can rewrite its formula as $m(x) = 7 \cdot x^{1/4}$. So $k = 7$ and $p = \frac{1}{4}$.

c. The function $l(x) = 3 \cdot 2^x$ is not a power function because the power is not constant. In fact, $l(x) = 3 \cdot 2^x$ is an exponential function.

d. Since

$$\sqrt{\frac{7}{x^5}} = \frac{\sqrt{7}}{\sqrt{x^5}} = \frac{\sqrt{7}}{x^{5/2}} = \sqrt{7} \cdot x^{-5/2}$$

we see that $s(x) = \sqrt{\frac{7}{x^5}}$ can be written in the form $y = kx^p$ where $k = \sqrt{7}$ and $p = -\frac{5}{2}$, so s is a power function.

As is the case with linear functions and exponential functions, given two points on the graph of a power function, we can find the function's formula.

EXAMPLE: Suppose that the points $(1, 81)$ and $(3, 729)$ are on the graph of a function f. Find an algebraic rule for f assuming that it is …

a. a linear function.

b. an exponential function.

c. a power function.

SOLUTIONS:

a. If f is a linear function we know that its rule has form $f(x) = mx + b$. We can use the two given points to solve for m.

$$\begin{aligned} m &= \frac{729 - 81}{3 - 1} \\ &= \frac{648}{2} \\ &= 324 \end{aligned}$$

So now we know that $f(x) = 324x + b$. We can use either one of the given points to find b. Let's use $(1, 81)$:

$$\begin{aligned} (1, 81) \quad &\Rightarrow \quad f(1) = 81 = 324(1) + b \\ &\Rightarrow \quad b = 81 - 324 \\ &\Rightarrow \quad b = -243 \end{aligned}$$

Thus, if f is linear, its rule is $f(x) = 324x - 243$.

b. If f is an exponential function we know its rule has form $f(x) = ab^x$. We can use the two given points to find two equations involving a and b:

$$\begin{aligned} (1, 81) \quad &\Rightarrow \quad f(1) = 81 = ab^1 \\ (3, 729) \quad &\Rightarrow \quad f(3) = 729 = ab^3. \end{aligned}$$

In Section III: Module 2 we solved similar systems of equations by forming ratios. Let's try a different method here: the *substitution* method.

Let's start by solving the first equation for a:

$$\begin{aligned} 81 &= ab^1 \\ \Rightarrow \quad a &= \frac{81}{b} \end{aligned}$$

Now we can substitute the expression $\frac{81}{b}$ for a in the second equation:

$$729 = ab^3$$
$$\Rightarrow 729 = \frac{81}{b} \cdot b^3$$
$$\Rightarrow 729 = 81 \cdot b^2$$
$$\Rightarrow \frac{729}{81} = b^2$$
$$\Rightarrow 9 = b^2$$
$$\Rightarrow b = \sqrt{9} = 3$$

(we don't need $\pm\sqrt{9}$ since the base of an exponential function is always positive)

Now that we know what b is, we can use the fact that $a = \frac{81}{b}$ to find a:

$$a = \frac{81}{b} = \frac{81}{3} = 27$$

Thus, if f is exponential, its rule is $f(x) = 27 \cdot 3^x$.

c. Since f is a power function we know that its rule has form $f(x) = kx^p$. We can use the two given points to find two equations involving k and p:

$$(1, 81) \Rightarrow f(1) = 81 = k(1)^p$$
$$(3, 729) \Rightarrow f(3) = 729 = k(3)^p.$$

We can use the first equation to immediately find k.

$$81 = k(1)^p$$
$$\Rightarrow k = 81$$

Now we can find p by substituting $k = 81$ into the second equation:

$$729 = 81(3)^p$$
$$\Rightarrow \frac{729}{81} = 3^p$$
$$\Rightarrow 9 = 3^p$$

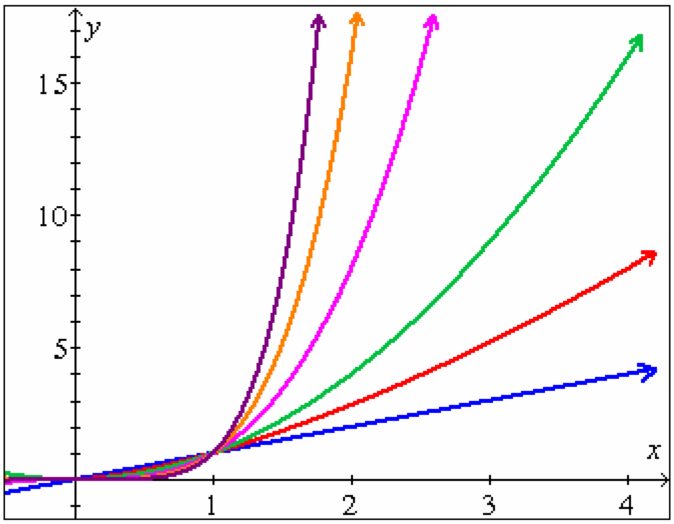
(note that this could be solved with logarithms if the solution weren't so obvious)

$$\Rightarrow p = 2$$

Thus, if f is a power function, its rule is $f(x) = 81x^2$.

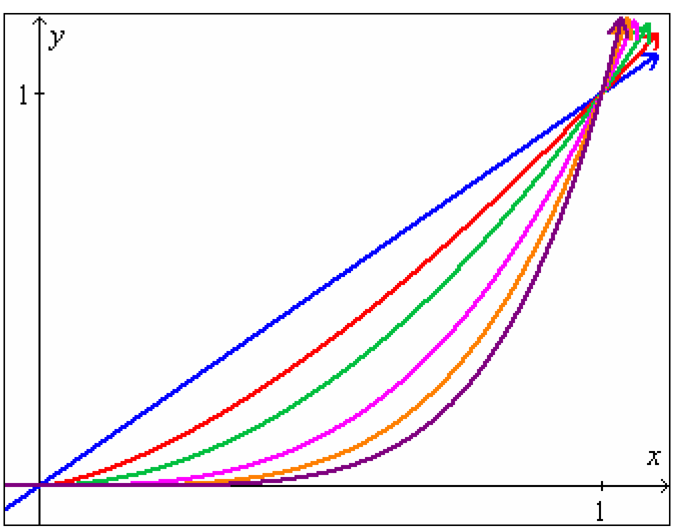
Graphs of Power Functions

For a power function $y = kx^p$ the greater the power of p, the faster the outputs grow. Below are the graphs of six power functions. Notice that as the power increases, the outputs increase more and more quickly. As x increases without bound (written "$x \to \infty$"), higher powers of x get a lot larger than (i.e., *dominate*) lower powers of x. (Note that we are discussing the **long-term** behavior of the function.)



The graphs of $y = x$, $y = x^{3/2}$, $y = x^2$, $y = x^3$, $y = x^4$, $y = x^5$.

As x approaches zero (written "$x \to 0$"), the story is completely different. If x is between 0 and 1, x^3 is larger than x^4, which is larger than x^5. (Try $x = 0.1$ to confirm this). For values of x near zero, *smaller* powers dominate. On the graph below, notice how on the interval $(0, 1)$ the linear power function $y = x$ dominates power functions of larger power.



The graphs of $y = x$, $y = x^{3/2}$, $y = x^2$, $y = x^3$, $y = x^4$, $y = x^5$.

EXAMPLE: Use your graphing calculator to graph $f(x) = 1000x^3$ and $g(x) = x^4$ for $x > 0$. Compare the long-term behavior of these two functions.

CLICK HERE (Be sure to turn up the volume on your computer!)

Could the graphs of $f(x) = 1000x^3$ and $g(x) = x^4$ intersect again for some value of $x > 1000$? To determine where these graphs intersect, let's solve the equation $f(x) = g(x)$:

$$f(x) = g(x)$$

$$1000x^3 = x^4$$

$$0 = x^4 - 1000x^3$$

$$0 = x^3(x - 1000).$$

Since the only solutions to this equation are $x = 0$ and $x = 1000$, the graphs of $f(x) = 1000x^3$ and $g(x) = x^4$ only intersect at $x = 0$ and $x = 1000$, so they do not intersect when $x > 1000$.

EXAMPLE: Use your graphing calculator to graph the power function $f(x) = x^3$ and the exponential function $g(x) = 2^x$ for $x > 0$. Compare the long-term behavior of these two functions.

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Key Point: Any positive increasing exponential function eventually grows faster than *any* power function.
