SECTION 5.2: VERIFYING TRIG IDENTITIES

PART A: EXAMPLE; STRATEGIES AND "SHOWING WORK"

One Example; Three Solutions

Verify the identity: $\dfrac{\csc\theta + \cot\theta}{\tan\theta + \sin\theta} = \cot\theta\csc\theta$.

Strategies and "Showing Work"

To verify an identity like this one, use the Fundamental Identities and algebraic techniques to simplify the side with the more complicated expression step-by-step until we end up with the expression on the other side. You may think of this as a simplification problem where the "answer" is given to you. The "answer" may be thought of as the top of a jigsaw puzzle box, the **TARGET** that you are aiming for. This is a strategy to keep in mind as you perform your manipulations.

Warning: Instructors generally want their students to "show work." In the simplification or verification process, you should probably write a new expression every time you apply a Fundamental Identity and every time you execute a "major" algebraic step (this may be a matter of judgment). If you are applying Fundamental Identities to different pieces of an expression, you may be able to apply them simultaneously in one step, provided that it is clear how and where they are being applied.

In this class, you will typically not be required to write the names of the various identity types you are using, but they will often be written in solutions for your reference.

The left-hand side (LHS) seems more complicated in this example, so we will operate on it until we obtain the right-hand side (RHS). In principle, you could begin with the RHS, or you could even work on both sides simultaneously until you "meet" somewhere in the middle. Some instructors may object to the latter method, however, perhaps because it may seem "sloppy." Even then, it could still inspire a more linear approach.

There are often different "good" approaches to problems such as these. You don't necessarily have to agree with your book's solutions manual!

Solution (Method 1)

(This may be the least efficient approach, though.)

Remember, we want to verify: $\dfrac{\csc\theta+\cot\theta}{\tan\theta+\sin\theta}=\cot\theta\csc\theta$

$$\frac{\csc\theta+\cot\theta}{\tan\theta+\sin\theta} = \frac{\dfrac{1}{\sin\theta}+\dfrac{\cos\theta}{\sin\theta}}{\dfrac{\sin\theta}{\cos\theta}+\sin\theta} \quad \text{(Reciprocal and Quotient Identities)}$$

We are breaking things down into expressions involving $\sin\theta$ and $\cos\theta$. They are like common currencies.

We can begin by simplifying the numerator ("N") and the denominator ("D") individually.

Tip: It may help to express $\sin\theta$ as $\dfrac{\sin\theta}{1}$.

$$= \frac{\dfrac{1+\cos\theta}{\sin\theta}}{\dfrac{\sin\theta}{\cos\theta}+\dfrac{\sin\theta}{1}}$$

← We already had a common denominator.

← $\cos\theta$ will be our common denominator.

$$= \frac{\dfrac{1+\cos\theta}{\sin\theta}}{\dfrac{\sin\theta}{\cos\theta}+\dfrac{\sin\theta}{1}\cdot\dfrac{\cos\theta}{\cos\theta}}$$

← We "build up" a fraction so that we have a common denominator.

$$= \frac{\dfrac{1+\cos\theta}{\sin\theta}}{\dfrac{\sin\theta}{\cos\theta}+\dfrac{\sin\theta\cos\theta}{\cos\theta}}$$

← We now have a common denominator. Add the fractions.

$$= \frac{\dfrac{1+\cos\theta}{\sin\theta}}{\dfrac{\sin\theta+\sin\theta\cos\theta}{\cos\theta}}$$

When we divide by a fraction, we are really multiplying by its reciprocal.

$$= \frac{1+\cos\theta}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta+\sin\theta\cos\theta}$$

We can factor the denominator of the second fraction, and we can perform a cancellation.

Tip: It often helps to consider easier factoring problems from Algebra I. If you have difficulty factoring $\sin\theta+\sin\theta\cos\theta$, try factoring $x+xy$. If you can see that $x+xy=(x)(1+y)$, then you should be able to see that $\sin\theta+\sin\theta\cos\theta=(\sin\theta)(1+\cos\theta)$

Tip: Grouping symbols can be very helpful when used appropriately, even when books don't use them as often!

$$= \frac{\cancel{1+\cos\theta}^{\,1}}{\sin\theta}\cdot\frac{\cos\theta}{(\sin\theta)\,\cancel{(1+\cos\theta)}^{\,1}}$$

$$= \frac{1}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta}$$

Keep the TARGET in mind. We are very close!
We will apply the Reciprocal and Quotient Identities to condense our expressions. (At the beginning, we used them to expand.)

$$= \csc\theta\cot\theta$$

Don't forget that multiplication of real quantities is commutative.
This strategy is sometimes overlooked by students!

Warning: Your final expression must look **exactly** like the TARGET.

$$= \cot\theta\csc\theta$$

Solution (Method 2)

Remember, we want to verify: $\dfrac{\csc\theta+\cot\theta}{\tan\theta+\sin\theta}=\cot\theta\csc\theta$

$$\frac{\csc\theta+\cot\theta}{\tan\theta+\sin\theta} = \frac{\dfrac{1}{\sin\theta}+\dfrac{\cos\theta}{\sin\theta}}{\dfrac{\sin\theta}{\cos\theta}+\sin\theta} \quad \text{(Reciprocal and Quotient Identities)}$$

We will multiply the N and the D by the least common denominator (LCD) of the overall fraction. The LCD is $\sin\theta\cos\theta$.

Warning: People often fail to properly apply the Distributive Property, so grouping symbols may help here! Also, it may help to express $\sin\theta$ as $\dfrac{\sin\theta}{1}$.

$$= \frac{\left(\dfrac{1}{\sin\theta}+\dfrac{\cos\theta}{\sin\theta}\right)}{\left(\dfrac{\sin\theta}{\cos\theta}+\dfrac{\sin\theta}{1}\right)}\cdot\frac{\sin\theta\cos\theta}{\sin\theta\cos\theta}$$

Warning: Instead of crossing things out (which is very risky if you have not yet applied the Distributive Property), you may want to cover up other expressions as you multiply things together. When in doubt, carefully write the step where you apply the Distributive Property, and then perform cancellations:

$$= \frac{\left(\frac{1}{\cancel{\sin\theta}} \cdot \cancel{\sin\theta}\cos\theta\right) + \left(\frac{\cos\theta}{\cancel{\sin\theta}} \cdot \cancel{\sin\theta}\cos\theta\right)}{\left(\frac{\sin\theta}{\cancel{\cos\theta}} \cdot \sin\theta\cancel{\cos\theta}\right) + \left(\frac{\sin\theta}{1} \cdot \sin\theta\cos\theta\right)}$$

Warning: "Wishful thinking" tends to creep into these problems involving cancellations in compound fractions. Remember that $\cos\theta$ multiplied by itself is $\cos^2\theta$. Many people incorrectly attempt to cancel and write "1."

$$= \frac{\cos\theta + \cos^2\theta}{\sin^2\theta + \sin^2\theta\cos\theta}$$

Factor the N and the D, and cancel common factors.

$$= \frac{(\cos\theta)\,\cancelto{1}{(1+\cos\theta)}}{(\sin^2\theta)\,\cancelto{1}{(1+\cos\theta)}} \qquad \begin{array}{l} \leftarrow \text{Think: } x + x^2 = (x)(1+x) \\ \hline \leftarrow \text{Think: } y^2 + y^2x = (y^2)(1+x) \end{array}$$

Keep the TARGET in mind. We may employ a "peeling" strategy. Remember that $\sin^2\theta = (\sin\theta)(\sin\theta)$, just as $y^2 = (y)(y)$.

$$= \frac{\cos\theta}{\sin\theta} \cdot \frac{1}{\sin\theta}$$

Finally, apply the Reciprocal and Quotient Identities to condense.

$$= \cot\theta\csc\theta$$

Solution (Method 3)

Remember, we want to verify: $\dfrac{\csc\theta+\cot\theta}{\tan\theta+\sin\theta}=\cot\theta\csc\theta$

$$\frac{\csc\theta+\cot\theta}{\tan\theta+\sin\theta}=\frac{\dfrac{1}{\sin\theta}+\dfrac{1}{\tan\theta}}{\tan\theta+\sin\theta}\quad\text{(Reciprocal Identities only)}$$

We could multiply the N and the D by the LCD, $\sin\theta\tan\theta$.

It turns out to be easier to first express the N as a "simple" fraction. Our LCD in the N is, again, $\sin\theta\tan\theta$.

$$=\frac{\dfrac{1}{\sin\theta}\cdot\dfrac{\tan\theta}{\tan\theta}+\dfrac{1}{\tan\theta}\cdot\dfrac{\sin\theta}{\sin\theta}}{\tan\theta+\sin\theta}\quad\leftarrow\text{Build up both fractions in the N.}$$

$$=\frac{\dfrac{\tan\theta+\sin\theta}{\sin\theta\tan\theta}}{\tan\theta+\sin\theta}$$

We may cancel the D and the "N of the N."

$$=\frac{\dfrac{\cancel{\tan\theta+\sin\theta}^{\,1}}{\sin\theta\tan\theta}}{{}_1\cancel{\tan\theta+\sin\theta}}$$

$$=\frac{1}{\sin\theta\tan\theta}$$

Let's "peel apart" (actually, factor) the one fraction as a product of two fractions.

$$= \frac{1}{\sin\theta} \cdot \frac{1}{\tan\theta}$$

Now, apply the Reciprocal Identities to condense.

$$= \csc\theta \cot\theta$$

$$= \cot\theta \csc\theta$$

PART B: "TRIG CONJUGATES"

When we rationalize the D in $\frac{1}{\sqrt{3}+\sqrt{5}}$, we multiply the N and the D by the conjugate of the D, $\sqrt{3}-\sqrt{5}$. This led to squarings that eliminated radicals in the D.

Similarly, when we rationalize (Think "real"-ize) the D in $\frac{1}{3+2i}$, we multiply the N and the D by the complex conjugate of the D, $3-2i$. This led to squarings that eliminated i in the D.

Similarly, we can use "trig conjugates" (not a standard term) to help us simplify, and verify identities involving, fractional expressions, particularly when the resulting squarings lead to expressions that we can attack using the Pythagorean Identities.

Example

Verify the identity: $\dfrac{1}{\sec\alpha - \tan\alpha} = \sec\alpha + \tan\alpha$.

Solution

Begin with the LHS, and multiply the N and the D of the LHS by the trig conjugate of the D, $\sec\alpha + \tan\alpha$.

Warning: Write the LHS (exactly) as your first expression, even if your first manipulation seems straightforward.

$$\frac{1}{\sec\alpha - \tan\alpha} = \frac{1}{(\sec\alpha - \tan\alpha)} \cdot \frac{(\sec\alpha + \tan\alpha)}{(\sec\alpha + \tan\alpha)}$$

For the new D, we will use the algebra rule: $(A-B)(A+B) = A^2 - B^2$

$$= \frac{\sec\alpha + \tan\alpha}{\sec^2\alpha - \tan^2\alpha}$$

The Pythagorean Identities may or may not help us now. It turns out that they do. Observe that the Pythagorean Identity $\tan^2\alpha + 1 = \sec^2\alpha$ may be rewritten as: $1 = \sec^2\alpha - \tan^2\alpha$.

$$= \frac{\sec\alpha + \tan\alpha}{1}$$

$$= \sec\alpha + \tan\alpha$$

Controversial Solution

Remember, we want to verify: $\dfrac{1}{\sec\alpha - \tan\alpha} = \sec\alpha + \tan\alpha$

We will treat the proposed identity as an equation. We will write a sequence of equivalent equations until we obtain an identity that we know to be true.

$$\frac{1}{\sec\alpha - \tan\alpha} = \sec\alpha + \tan\alpha$$

We will multiply both sides by $\sec\alpha - \tan\alpha$. For the purposes of verifying the identity, we may assume that both $\sec\alpha$ and $\tan\alpha$ are defined.
We may also assume that $\sec\alpha - \tan\alpha \neq 0$; otherwise, the LHS would be undefined.

$$\begin{aligned} 1 &= \left(\sec\alpha + \tan\alpha\right)\left(\sec\alpha - \tan\alpha\right) \\ 1 &= \sec^2\alpha - \tan^2\alpha \\ \tan^2\alpha + 1 &= \sec^2\alpha \end{aligned}$$

The last equation is a known Pythagorean Identity.

Although the author is not particularly bothered by this method, it does bother many other instructors, and it will be discouraged. Always follow your instructor's cue, and ask him/her about "good form and procedure" if you are unsure.

Although we tend to disregard domain issues when doing these kinds of problems, we must be very careful about potentially multiplying or dividing both sides of an equation by a quantity that is 0 or undefined. This may be a key reason for the controversy surrounding this method. Addition and subtraction tend to be less controversial operations, as are multiplication and division by nonzero constants. When in doubt, keep domain issues in mind!

PART C: A SUMMARY OF STRATEGIES

This list is not intended to be comprehensive, but it is a nice toolbox!

1) Longer $\rightarrow$ Shorter

We usually want to start with the "longer" (i.e., the more complicated) side and try to get to the "shorter" side. You could tinker with the "shorter" side as necessary as you strategize, or you could re-express it outright.

2) TARGET

Keep the TARGET (the expression you're aiming for) in mind.
This can influence strategies.

3) Fundamental Identities

Keep all the Fundamental Identities in mind.

4) LCDs

Use LCDs for adding and subtracting fractions and for simplifying compound fractions.

5) Trig Conjugates

Consider using trig conjugates in conjunction with Pythagorean Identities, especially when pairs of trig functions found in Pythagorean Identities (sin and cos, tan and sec, cot and csc), 1, and/or -1 are involved.

“DECOMPOSITION” STRATEGIES

6) Go to sin and cos

Consider using the Reciprocal and Quotient Identities to break everything down into expressions involving sin and cos.

7) Factoring

Cancellations may result. Pythagorean Identities may be useful.

8) Splitting a Fraction (Multiplication and Division): “Peeling”

This is like a basic form of factoring. For example, see Notes 5.14:

$$\frac{\cos\theta}{\sin^2\theta} = \frac{\cos\theta}{\sin\theta} \cdot \frac{1}{\sin\theta}$$

Keep the TARGET in mind.

9) Splitting a Fraction (Addition and Subtraction): Splitting a Fraction through the N (Numerator)

For example, you may use the template: $\frac{x+y}{z} = \frac{x}{z} + \frac{y}{z}$

Warning: Remember that we cannot split through the D (Denominator) in a similar fashion.

OTHER STRATEGIES

10) **Looking at a similar problem in Algebra I, which we did in 9), may help you. Also, remember how to manipulate fractions back in Arithmetic. For example, when you divide by a fraction, you are really multiplying by its reciprocal.**

11) **In general, be neat, and show work, especially when you are applying the Fundamental Identities and algebraic strategies.**