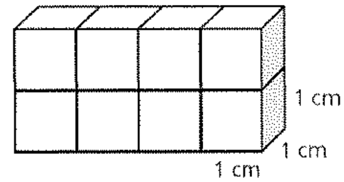
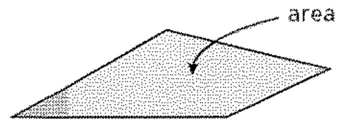
4.4.D1 VOLUME OF PRISMS & CYLINDERS

Recall that the *volume* of a three-dimensional figure is the number of non-overlapping cubic units contained in the interior of the figure. For example, the prism at right has a volume of 8 cubic centimeters. You can use this idea to develop volume formulas.



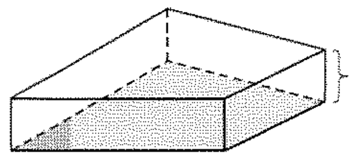
PROBLEM 1 ~ Developing a Basic Volume Formula

A Consider a figure that is the base of a prism or cylinder. Assume the figure has an area of B square units.



B Use the base to build a prism or cylinder with height 1 unit. $h=1$

This means the prism or cylinder contains $\underline{B}$ cubic units.

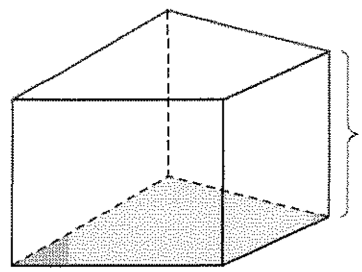


C Now use the base to build a prism or cylinder with a height of h units.

The volume of this prism or cylinder must be $\underline{h}$ times the volume of the prism or cylinder whose height is 1 unit.

So, the volume of the prism or cylinder is $\underline{Bh}$ cubic units.

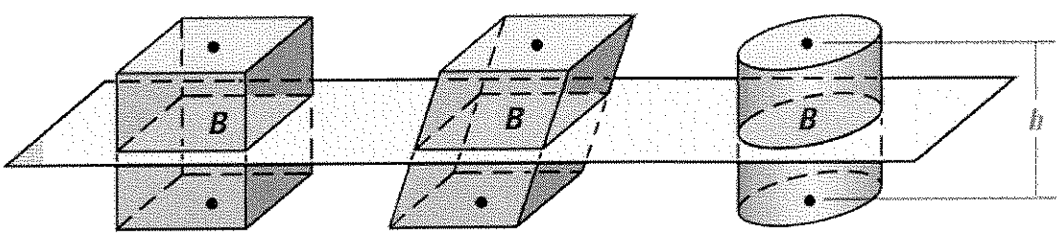
Volume formula for prisms



RECALL ~ Cavalieri's Principle

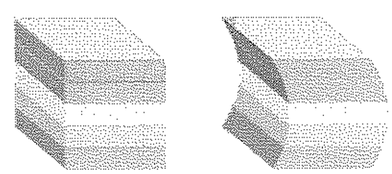
If two solids have the same height and the same cross-sectional area at every level, then they have the same volume.

Consider the solids below. All three have cross sections with equal areas, B, and all three have equal heights, h. By Cavalieri's Principle, it follows that each solid has the same volume.



EXAMPLE 1 ~ Cavalieri's Principle

Each stack of memo papers shown contains 500 sheets of paper. Explain why the stacks have the same volume. Then calculate the volume, given that each sheet of paper is 3 inches by 3 inches by 0.01 inches.



the stacks have the same volume b/c each paper has the same area: $B = 3 \cdot 3 = 9$ in^2 AND the same # of sheets: 500

$V = Bh \cdot 500$

$V = 9 \cdot 0.01 \cdot 500$

$V = 45$ in^3

Volume of a Cylinder

The volume V of a cylinder with base area B and height h is given by $V = Bh$ (or $V = \pi r^2 h$, where r is the radius of the base).

EXAMPLE 2 ~ Comparing Densities

You gather data about two wood logs that are approximately cylindrical. Based on the data in the table, which wood is denser, Douglas fir or American redwood?

Type of Wood	Diameter (ft)	Height (ft)	Weight (lb)
Douglas fir	1 0.5	6	155.5
American redwood	3 1.5	4	791.7

a. Find the volume of the Douglas fir log.

$V = \pi r^2 h = \pi \cdot 0.5^2 \cdot 6 = 1.5\pi \approx 4.712 \text{ ft}^3$

b. Find the volume of the American redwood log.

$V = \pi r^2 h = \pi \cdot 1.5^2 \cdot 4 = 9\pi \approx 28.274 \text{ ft}^3$

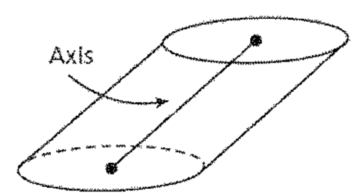
c. Calculate and compare densities.

Douglas fir $= \dfrac{155.5}{1.5\pi} \approx 33 \text{ lb/ft}^3$ *density = $\dfrac{\text{weight}}{\text{unit volume}}$

American redwood $= \dfrac{791.7}{9\pi} \approx 28 \text{ lb/ft}^3$

the Douglas fir is denser than the American redwood.

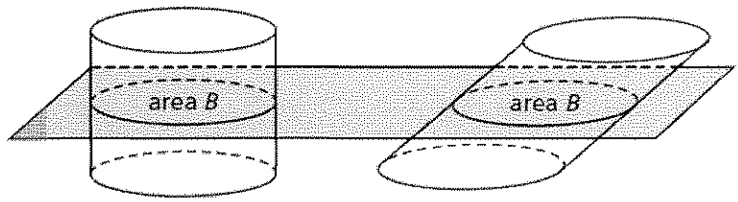
The axis of a cylinder is the segment whose endpoints are the centers of the bases. A right cylinder is a cylinder whose axis is perpendicular to the bases. An **oblique cylinder** is a cylinder whose axis is not perpendicular to the bases. Cavalieri's principle makes it possible to extend the formula for the volume of a cylinder to oblique cylinders.



Oblique cylinder

Cavalieri's Principle

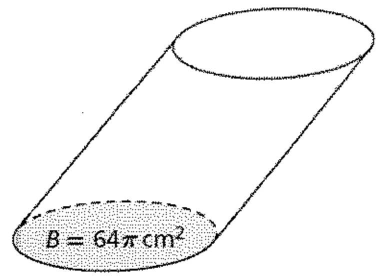
If two solids have the same height and the same cross-sectional area at every level, then the two solids have the same volume.



You can think of any oblique cylinder as a right cylinder that has been "pushed over" so that the cross sections at every level have equal areas. By Cavalieri's principle, the volume of an oblique cylinder is equal to the volume of the associated right cylinder. This means the formula $V = Bh = \pi r^2 h$ works for any cylinder.

EXAMPLE 3 ~ Finding the Volume of an Oblique Cylinder

The height of the cylinder shown here is twice the radius. What is the volume of the cylinder? Round to the nearest tenth.



a. Find the height of the cylinder. To do so, first find the radius of the cylinder.

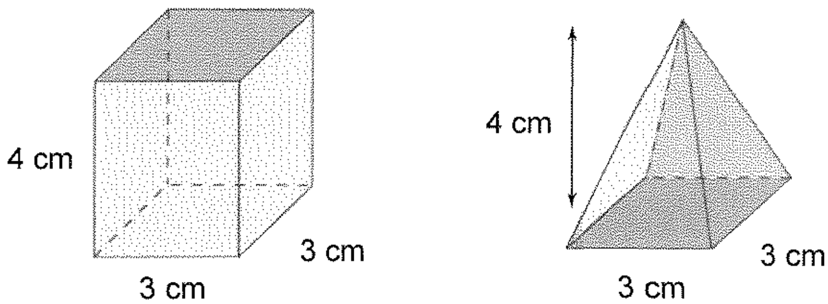
$B = \pi r^2 = 64\pi$
$r^2 = 64$
$r = 8$ $\quad h = 2r = 2 \cdot 8 = 16$ cm

b. Find the volume of the cylinder.

$V = \pi r^2 h$ OR $V = Bh$
$V = 64\pi \cdot 16 = 1024\pi$
$V \approx 3217.0$ cm^3

4.4.D2 VOLUME OF PYRAMIDS & CONES

PROBLEM 1 ~ Connecting the Volumes of Prisms and Pyramids



a. How are the solids alike? same height, base is a square w/ same side lengths

b. How are they different? prism vs. pyramid

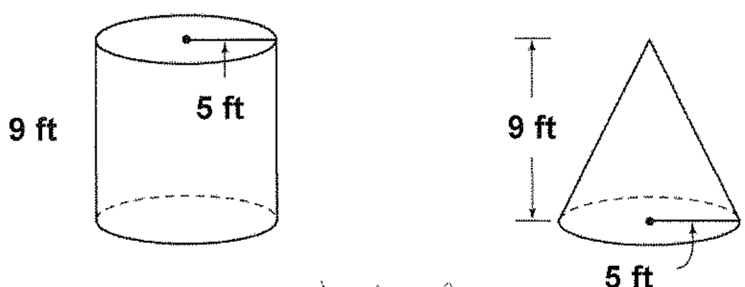
c. Find the volume of the rectangular prism.

$V = lwh = 3^2 \cdot 4 = 36\ cm^3$

d. How does the volume of the prism compare to that of the rectangular pyramid, if the pyramid's volume is 12 cubic centimeters? $36 = 3 \cdot 12$

the prism is 3xs the volume of the pyramid OR the pyramid is 1/3 the volume of the prism

PROBLEM 2 ~ Connecting the Volumes of Cylinders and Cones



a. How are the solids alike? Same height; circular base w/ the same radius

b. How are they different? cylinder vs. cone

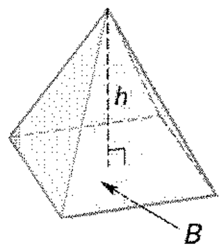
c. Find the volume of the cylinder.

$V = \pi r^2 h = \pi \cdot 5^2 \cdot 9 = 225\pi\ ft^3$

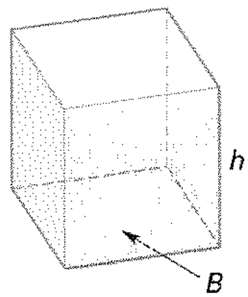
d. How does the volume of the cylinder compare to that of the cone, if the cone's volume is 75π cubic feet? $225\pi = 3 \cdot 75\pi$

the cylinder is 3xs the volume of the cone OR the cone is 1/3 the volume of the cylinder

VOLUME OF PRISMS & PYRAMIDS

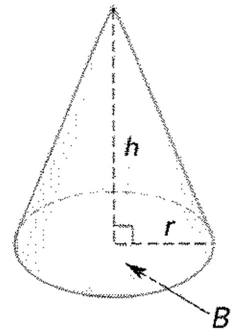


$V = \frac{1}{3}Bh$

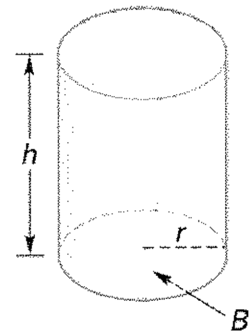


$V = Bh$

VOLUME OF CYLINDERS & CONES



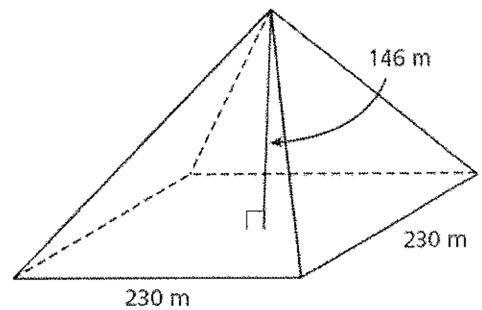
$V = \frac{1}{3}Bh$
$= \frac{1}{3}\pi r^2 h$



$V = Bh$
$= \pi r^2 h$

EXAMPLE 1 ~ Volume of Pyramids

The Great Pyramid in Giza, Egypt, is approximately a square pyramid with the dimensions shown. The pyramid is composed of stone blocks that are rectangular prisms. An average block has dimensions 1.3 meters by 1.3 meters by 0.7 meters. Approximately how many stone blocks were used to build the pyramid?



a. Find the volume of the pyramid.

$V = \frac{1}{3}Bh = \frac{1}{3} \cdot 230^2 \cdot 146 \approx 2574466.\overline{6}$

b. Find the volume of an average block.

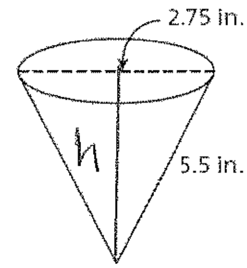
$V = lwh = 1.3^2 \cdot 0.7 = 1.183\ m^3$

c. Find the approximate number of stone blocks in the pyramid.

$\#\text{ of blocks} = \frac{V_{pyr.}}{V_{block}} \approx 2176218.653$

2176219 blocks

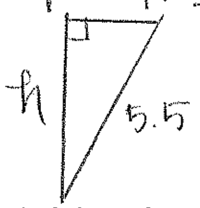
EXAMPLE 2 ~ **Volume of Cones**



A conical paper cub has the dimensions shown. How many fluid ounces of liquid does the cup hold? Round to the nearest tenth. (*Hint:* 1 in.^{3} ≈ 0.554 fl oz.)

a. Find the radius and height of the cone.

$r = 1.375$



$c^2 = a^2 + b^2$

$5.5^2 = h^2 + 1.375^2$

$30.25 = h^2 + 1.890625$

$28.359375 = h^2$

$h \approx 5.325$ in

b. Find the volume of the cone to the nearest hundredth.

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \cdot 1.375^2 \cdot 5.325$

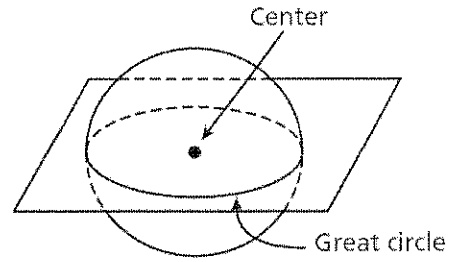
$V \approx 10.54$ in^3

c. Convert the volume to fluid ounces.

$$\frac{1 \text{ in}^3}{10.54 \text{ in}^3} = \frac{0.554 \text{ fl oz.}}{x}$$

5.84 fl. oz.

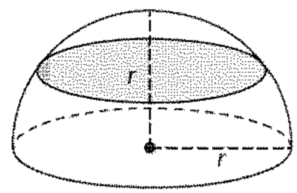
4.5 VOLUME OF A SPHERE

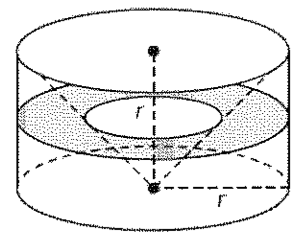


Recall that a *sphere* is the set of points in space that are a fixed distance from a point called the *center* of the sphere. The intersection of a sphere and a plane that contains the center of the sphere is a *great circle*. A great circle divides a sphere into to congruent halves that are called *hemispheres*.

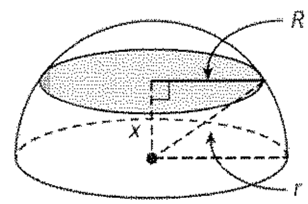
PROBLEM 1 ~ Developing a Volume Formula

Plan: To find the volume of a given sphere, consider a hemisphere of the sphere and a cylinder with the same radius and height as the hemisphere from which a cone has been removed. Show that the two solids have the same cross-sectional area at every level and apply Cavalieri's principle to conclude that the figures have the same volume.





a. To show that cross sections have the same area at every level, consider cross sections at a distance of x above the base, as shown.



The cross section of the hemisphere is a disc. Use the Pythagorean Theorem to write a relationship among $r, x,$ & R.

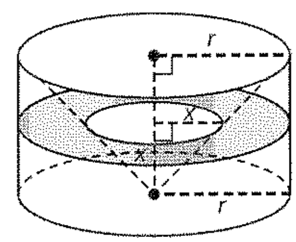
$x^2 + R^2 = r^2$

$R^2 = r^2 - x^2$

Solving for R gives $R =$ $\sqrt{r^2 - x^2}$

So, the area of the cross-sectional disc is πR^2 or $\pi(r^2 - x^2)$

b. The cross section of the cylinder with the cone removed is a ring. To find the area of the ring, find the area of the outer circle and subtract the area of the inner circle.



The outer circle has radius r, so its area is πr^2.

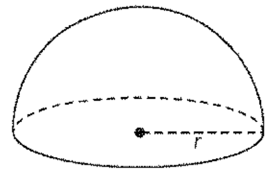
The figure includes a pair of isosceles right triangles that are similar. This makes it possible to find the radius of the inner circle.

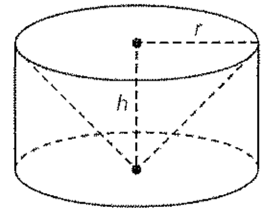
The inner circle has radius x, so its area is πx^2.

So, the area of the cross-sectional ring is $\pi r^2 - \pi x^2 = \pi(r^2 - x^2)$

By the distributive property, the areas of the cross sections are equal.

c. By Cavalieri's Principle, the hemisphere has the same volume as the cylinder with the cone removed.





V(hemisphere)= V(cylinder) – V(cone)

$$= \pi r^2 h - \frac{1}{3}\pi r^2 h$$

$$= \frac{2}{3}\pi r^2 h$$

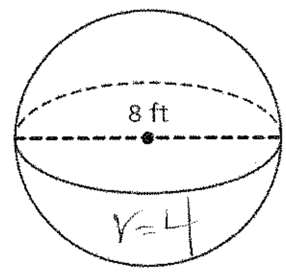
* the height of the sphere is equal to the radius

$$= \frac{2}{3}\pi r^3$$

The volume of the sphere is twice the volume of the hemisphere. $2 \cdot \frac{2}{3}\pi r^3$

So, the volume of the sphere is $\frac{4}{3}\pi r^3$.

EXAMPLE 1 ~ Solving a Volume Problem



A British thermal unit (BTU) is a unit of energy. It is approximately the amount of energy needed to increase the temperature of one pound of water by one degree Fahrenheit. As you will see in the following example, the energy content of a fuel may be measured in BTUs per unit of volume.

A spherical gas tank has the dimensions shown. When filled with natural gas, it provides 275,321 BTU. How many BTUs does one cubic foot of natural gas yield? Round to the nearest BTU.

a. Find the volume of the sphere.

$$\frac{4}{3}\pi \cdot 4^3 = \frac{256\pi}{3} \text{ ft}^3 \approx$$

b. Find the number of BTUs contained in one cubic foot of natural gas.

$$275321 \div \frac{256\pi}{3} \approx 1027 \text{ BTUs}$$