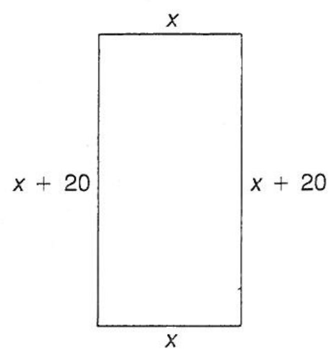
EXAMPLE 5 Dean uses 160 feet of fence to enclose his rectangular garden plot. If the length is 20 feet more than the width, find the dimensions of the plot.

Solution We begin by letting x denote the width. Rather than introduce another variable for the length we note that the length is then $x + 20$. See Figure 2.1.

FIGURE 2.1



The perimeter is given by $2x + 2(x + 20)$. Thus we have the equation

$$\begin{aligned} 2x + 2(x + 20) &= 160 \\ 2x + 2x + 40 &= 160 && \text{(Distributive property)} \\ 4x + 40 &= 160 && \text{(Collect terms)} \\ 4x &= 120 && \text{(Subtract 40 from both sides)} \\ x &= 30 && \text{(Divide both sides by 4)} \\ x + 20 &= 50 \end{aligned}$$

The garden plot is 30 feet wide by 50 feet long. ◆

EXAMPLE 6 Ms. Pleiman sells an investment property and places the proceeds in certificates of deposit (CDs). The maximum current interest rate is paid by a 1-year certificate yielding 10 percent. She can lock in an 8 percent rate for $2\frac{1}{2}$ years. Thinking that interest rates may go down, she splits her money, depositing three times as much in the 10 percent account as in the 8 percent account. The yearly return on these deposits is $11,780. How much did she realize from the sale of her property? How much did she invest in each account?

Solution Let x = amount (in dollars) invested at 8 percent

Then $3x$ = amount (in dollars) invested at 10 percent

and $4x$ = amount (in dollars) realized from sale of the property.

It is helpful to display the given information as follows.

Amount invested ·	Interest rate =	Yearly return
x	8%	$(0.08)x$
$3x$	10%	$(0.10)3x$

For each year's income, we have the following equation:

$$\begin{array}{ccccc} \text{(Interest from} & + & \text{(Interest from} & = & \text{(Total interest)} \\ \text{8 percent account)} & & \text{10 percent account)} & & \\ (0.08)x & + & (0.10)3x & = & 11{,}780 \end{array}$$

We then solve for the unknowns.

$$\begin{aligned} (0.08)x + (0.30)x &= 11{,}780 \\ 8x + 30x &= 1{,}178{,}000 && \text{(Multiply both sides by 100)} \\ 38x &= 1{,}178{,}000 \\ x &= 31{,}000 && \text{(Divide both sides by 38)} \\ 3x &= 93{,}000 \\ 4x &= 124{,}000 \end{aligned}$$

Ms. Pleiman realized \$124,000 from the sale of her property. She then deposited \$31,000 at 8 percent interest and \$93,000 at 10 percent interest. ◆

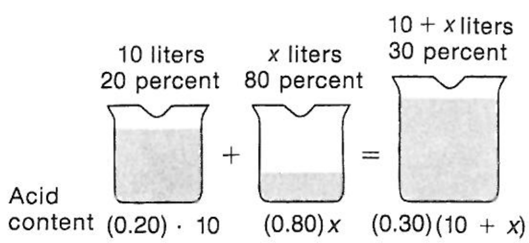
EXAMPLE 7 A chemist needs to mix 10 liters of a 20 percent acid solution with an 80 percent acid solution to obtain a mixture that will be a 30 percent acid solution. How many liters of the 80 percent acid solution must be used?

Solution Let

x = number of liters of 80 percent solution that must be added to 10 liters of 20 percent solution to yield a 30 percent solution

The total amount of acid in each container is shown in Figure 2.2.

FIGURE 2.2



To find x, we must solve the equation

$$\begin{aligned} (0.20)\cdot 10 + (0.80)x &= (0.30)(10 + x) \\ 2 + 0.8x &= 3 + 0.3x \\ 0.8x - 0.3x &= 3 - 2 \\ 0.5x &= 1 \\ 5x &= 10 && \text{(Multiply both sides by 10)} \\ x &= 2 \end{aligned}$$

Thus, 2 liters of 80 percent solution must be added to 10 liters of 20 percent solution to yield a 30 percent solution. The formula for 30 percent solution is two parts 80 percent solution with ten parts 20 percent solution, or "strong and weak acids are mixed in the ratio 1 to 5." ◆

EXAMPLE 8 The Vacation Realty Company plans to sell shares in a luxury resort condominium. Each share will entitle the shareholder to use of the condominium for one week each year. The company already has 50 prospective buyers. Knowing that not everyone will want to return to this particular spot every year, the company advises the prospective buyers to allow the project to be oversold. By allowing 25 additional shares to be sold, the original investors can save \$1200 each. What is the total sale price of the condominium? What is the cost per share if 50 shares are sold? 75 shares?

Solution Let P = total sales price (in dollars) of the condominium

Then $\dfrac{P}{50}$ = cost per share with 50 investors

and $\dfrac{P}{75}$ = cost per share with 75 investors

With 75 investors, the cost per share is \$1200 less than with 50 investors, which gives us the equation

$$\text{(Cost with 50 investors)} - \$1200 = \text{(Cost with 75 investors)}$$

$$\frac{P}{50} - 1200 = \frac{P}{75}$$

We solve this equation for P as follows.

$$150 \cdot \left(\frac{P}{50} - 1200\right) = \frac{P}{75} \cdot 150$$

$$3P - 180{,}000 = 2P$$

$$P = 180{,}000$$

Thus $$\frac{P}{50} = \frac{180{,}000}{50} = 3600$$

and $$\frac{P}{75} = \frac{180{,}000}{75} = 2400$$

The condominium costs \$180,000; the costs per share are \$3600 or \$2400 depending on whether 50 or 75 shares are sold.

EXAMPLE 9 A small plane can fly 585 nautical miles with the wind in the same time that it takes to fly 495 nautical miles against the wind. If the speed of the wind is 10 knots (nautical miles per hour), how fast does the plane fly when there is no wind?

Solution Let x = speed of plane (no wind), in knots

Then $x + 10$ = speed of plane flying with the wind

and $x - 10$ = speed of plane flying against the wind

We use the formula

$$\text{distance} = \text{rate} \cdot \text{time} \quad \text{or} \quad d = rt$$

This formula can also be written as

$$\frac{\text{distance}}{\text{rate}} = \text{time} \quad \text{or} \quad \frac{d}{r} = t$$

It is convenient to display the given information in a table as follows.

	Distance /	Rate =	Travel time
With the wind	585	$x + 10$	$\dfrac{585}{x+10}$
Against the wind	495	$x - 10$	$\dfrac{495}{x-10}$

Since the travel times are said to be the same, we set them equal to each other.

$$\frac{585}{x+10} = \frac{495}{x-10}$$

$$\cancel{(x+10)}(x-10)\frac{585}{\cancel{(x+10)}} = \frac{495}{\cancel{(x-10)}}(x+10)\cancel{(x-10)}$$

$$585x - 5850 = 495x + 4950$$

$$(585 - 495)x = 5850 + 4950$$

$$90x = 10{,}800$$

$$x = 120$$

The speed of the plane is 120 knots when there is no wind. ◆

EXAMPLE 10 Two drainage pipes are used to empty a pool. If the pool can be drained in 12 hours using pipe A alone and it can be drained in 6 hours using pipe B alone, how long will it take to empty the pool if both pipes are used? (The answer is *not* 18 hours.)

Solution Let n be the number of hours required to drain the pool using both pipes together. We can display this information in a table as follows.

	Number of hours to drain pool	Part of pool drained in one hour
Pipe A	12	$\dfrac{1}{12}$
Pipe B	6	$\dfrac{1}{6}$
Both pipes	n	$\dfrac{1}{n}$

The two pipes together then drain

$$\frac{1}{12} + \frac{1}{6} = \frac{1}{n}$$

of the pool in one hour. We solve this equation for n as follows.

$$12n\left(\frac{1}{12}+\frac{1}{6}\right)=\frac{1}{n}\cdot 12n$$

$$\frac{12n}{12}+\frac{12n}{6}=\frac{12n}{n}$$

$$n+2n=12$$

$$3n=12$$

$$n=4$$

The pool can be emptied in only 4 hours using the two pipes together.

EXAMPLE 11 A painter knows that her assistant can paint only half as much in a day as she can. She is planning to fire the assistant, but she must finish painting the houses in a certain real estate development by a given date or forfeit a cash bond. In the past 5 days they have painted three houses working together. How long would it take each woman working alone to paint a house?

Solution Let t = number of days required for the painter to paint a house.

Then $2t$ = number of days required for the assistant to paint a house.

We set up the following table.

	Number of houses painted, H	/ Number of days, D	= Houses painted per day, H/D
Painter	1	t	$\frac{1}{t}$
Assistant	1	$2t$	$\frac{1}{2t}$
Together	3	5	$\frac{3}{5}$

Considering the number of houses painted per day, we obtain the following equation.

$$\frac{1}{t}+\frac{1}{2t}=\frac{3}{5}$$

$$\frac{3}{2t}=\frac{3}{5} \qquad \left(\frac{1}{t}=\frac{2}{2t}\right)$$

$$\frac{1}{2t}=\frac{1}{5} \qquad \text{(Divide both sides by 3)}$$

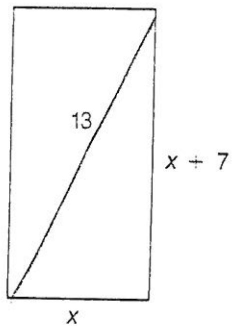
$$2t=5 \qquad \text{(Invert both sides)}$$

$$t=\frac{5}{2}=2\frac{1}{2}$$

The painter can paint a house in $2\frac{1}{2}$ days; her assistant takes 5 days.

EXAMPLE 2 A sidewalk is constructed diagonally across a rectangular park that is 700 yards longer than it is wide. If the sidewalk is 1300 yards long, what are the dimensions of the park?

FIGURE 2.4



Solution Label the dimensions (in hundreds of yards) as in Figure 2.4. Then by the Pythagorean theorem

$$x^2 + (x + 7)^2 = 13^2$$
$$x^2 + (x^2 + 14x + 49) = 169 \quad \text{(Formula 1, Section 1.6)}$$
$$2x^2 + 14x - 120 = 0 \quad \text{(Collect terms)}$$
$$x^2 + 7x - 60 = 0 \quad \text{(Divide through by 2)}$$
$$(x + 12)(x - 5) = 0 \quad \text{(Factor)}$$
$$x = 5, \ -12$$

(Again we discard $x = -12$ since we must have $x > 0$.)

The other side is $\quad x + 7 = 12$

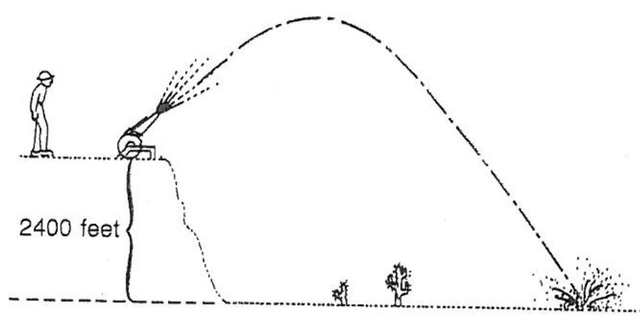
The dimensions of the park are 500 yards by 1200 yards.

EXAMPLE 3 The altitude h in feet of a projectile t seconds after firing is given by

$$h = -16t^2 + v_u t + h_0$$

where v_u is the *upward* component of the projectile's initial velocity in feet per second and h_0 is the altitude in feet from which the projectile is fired. A rocket is launched from a hilltop 2400 feet above the desert with an initial upward velocity of 400 feet per second. When will it land on the desert? (See Figure 2.5.)

FIGURE 2.5



Solution The altitude of the rocket t seconds after launching is

$$h = -16t^2 + 400t + 2400$$

When it lands, its altitude is zero. To determine the time when its altitude is zero, we must solve the quadratic equation

$$h = -16t^2 + 400t + 2400 = 0$$
$$-16(t^2 - 25t - 150) = 0 \quad \text{(Factor out } -16\text{)}$$
$$t^2 - 25t - 150 = 0 \quad \text{(Divide through by } -16\text{)}$$
$$(t - 30)(t + 5) = 0 \quad \text{(Factor)}$$
$$t = 30, \ -5$$

The solution $t = -5$ represents the time 5 seconds *before* launching and is irrelevant. Thus, the rocket should land on the desert 30 seconds after it is fired. ◆

EXAMPLE 4 If the rocket described in Example 3 is launched with an initial *downward* velocity of 80 feet per second, when does it land on the desert?

Solution Since v_u in the formula given in Example 3 represents *upward* velocity, we must set $v_u = -80$ to represent the *downward* velocity. Then we solve as follows:

$$h = -16t^2 - 80t + 2400 = 0$$
$$-16(t^2 + 5t - 150) = 0$$
$$(t - 10)(t + 15) = 0 \qquad \text{(Divide through by } -16 \text{ and factor)}$$
$$t = 10, \ -15$$

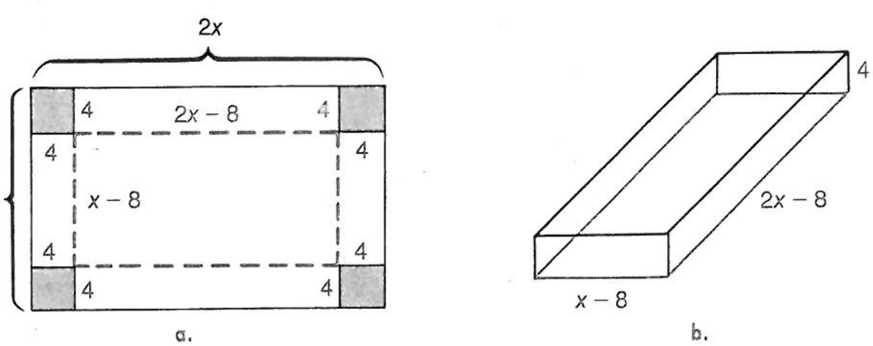
The rocket lands 10 seconds after firing (and *not* 15 seconds before firing). ◆

EXAMPLE 5 A rectangular piece of metal is to be used to make a box by cutting 4-inch squares from each of its corners and then folding up the sides. If the piece of metal is twice as long as it is wide and the volume of the box is 768 cubic inches, find the dimensions of the piece of metal.

Solution We label the width and length of the metal as x and $2x$, respectively. See Figure 2.6(a). Referring to Figure 2.6(b), we see that the volume of the box is

$$V = (2x - 8)(x - 8)4 = 768 \qquad (V = \text{length} \cdot \text{width} \cdot \text{height})$$
$$2(x - 4)(x - 8) = 192 \qquad \text{(Factor and divide through by 2)}$$
$$x^2 - 12x + 32 = 96$$
$$x^2 - 12x - 64 = 0 \qquad \text{(Subtract 96 from both sides)}$$
$$(x - 16)(x + 4) = 0 \qquad \text{(Factor)}$$
$$x = 16, \ -4$$

FIGURE 2.6



We can discard $x = -4$ since we must have $x > 0$. The dimensions of the metal are 16 inches by 32 inches (x and $2x$). ◆

EXAMPLE 6 Economists say that a given commodity is in equilibrium when supply equals demand. Suppose that when corn sells for \$1.50 per bushel, there is a demand for 100,000 metric tons and a supply of only 50,000 metric tons. Also, suppose that each x-cent increase in the price per bushel stimulates an additional production of $5x^2$ metric tons and decreases the demand by $750x$ metric tons. What is the equilibrium price for corn?

Solution At a price of $(150 + x)$ cents per bushel,

$$\text{Production} = 50{,}000 + 5x^2$$
$$\text{Demand} = 100{,}000 - 750x$$

Equating these two expressions for equilibrium:

$$\begin{aligned} 50{,}000 + 5x^2 &= 100{,}000 - 750x \\ 5x^2 + 750x - 50{,}000 &= 0 \\ x^2 + 150x - 10{,}000 &= 0 \qquad \text{(Divide through by 5)} \\ (x + 200)(x - 50) &= 0 \\ x &= -200,\ 50 \end{aligned}$$

The value $x = -200$ represents a decrease of \$2.00 in the price of corn to $-50¢$. Such a solution is meaningless. The value $x = 50$ makes sense. An equilibrium price for corn is $(150 + x)$ cents $= (150 + 50)$ cents $=$ \$2.00 per bushel. ◆

The supply-demand relationship is actually much more complicated than that given in Example 6. In this and other examples and exercises, simplified expressions are used so that the problem can be worked in this course.

EXAMPLE 7 On an outing, Cindy's Explorer troop rowed upstream 60 miles and returned. The entire trip took 40 hours. If the stream was flowing at the rate of 2 miles per hour, how fast could Cindy's troop row in still water? How much more time was spent rowing upstream than downstream?

Solution Let x = rate (in miles per hour) the troop rowed in still water

Then $x + 2$ = rate they traveled downstream

and $x - 2$ = rate they traveled upstream

Recall from Section 2.2 that

$$\text{distance} = \text{rate} \cdot \text{time} \qquad \text{or} \qquad \frac{\text{distance}}{\text{rate}} = \text{time}$$

We have the following information.

	Distance /	Rate =	Travel time
Upstream	60	$x - 2$	$\frac{60}{x-2}$
Downstream	60	$x + 2$	$\frac{60}{x+2}$

We then obtain the equation:

$$\begin{aligned}
(\text{Time upstream}) + (\text{Time downstream}) &= (\text{Total time})\\
\frac{60}{x-2} + \frac{60}{x+2} &= 40\\
(x-2)(x+2)\left[\frac{60}{x-2} + \frac{60}{x+2}\right] &= 40(x-2)(x+2)\\
60(x+2) + 60(x-2) &= 40(x^2-4)\\
60x + 120 + 60x - 120 &= 40x^2 - 160\\
-40x^2 + 120x + 160 &= 0\\
x^2 - 3x - 4 &= 0 \qquad \text{(Divide through by } -40)\\
(x-4)(x+1) &= 0\\
x &= 4, -1
\end{aligned}$$

The value $x = -1$ is meaningless in this problem since we must have $x \geq 0$. Thus, Cindy's troop could row 4 miles per hour in still water. They traveled 2 ($= x - 2$) miles per hour upstream and 6 ($= x + 2$) miles per hour downstream. The upstream trip took them 30 hours [$= 60/(x - 2)$] and the downstream trip took 10 hours [$= 60/(x + 2)$]. ◆

EXAMPLE 8 The $60,000 cost of a certain charter flight must be shared equally by those who take the trip. A certain number of travelers have already signed up for the trip. If 100 additional customers sign up, the cost to each of the original patrons will be reduced by $100. How many have already signed up?

Solution Let x = the number of people already signed up. Then we have the following information.

Cost /	Number of patrons =	Cost to each patron
$60,000	x	$\frac{60{,}000}{x}$
$60,000	$x + 100$	$\frac{60{,}000}{x+100}$

Thus, we obtain the equation

$$(\text{Original cost}) - \$100 = (\text{Cost if 100 more sign up})$$

$$\frac{60{,}000}{x} - 100 = \frac{60{,}000}{x + 100}$$

$$x(x + 100) \cdot \left[\frac{60{,}000}{x} - 100\right] = \left[\frac{60{,}000}{x + 100}\right] \cdot x(x + 100)$$

$$(x + 100)(60{,}000) - 100x(x + 100) = 60{,}000x$$

$$\cancel{60{,}000x} + 6{,}000{,}000 - 100x^2 - 10{,}000x = \cancel{60{,}000x}$$

$$-100x^2 - 10{,}000x + 6{,}000{,}000 = 0$$

$$x^2 + 100x - 60{,}000 = 0 \qquad \text{(Divide through by } -100\text{)}$$

$$(x + 300)(x - 200) = 0$$

$$x = \cancel{-300}, 200$$

We cannot have -300 people already signed up. Even though -300 is a solution of the *equation,* it is not a solution of the *problem.* Thus, 200 people have already signed up. ◆

Section 2.5 Exercises

1. Find two consecutive positive even integers, the product of which is 168.

2. Find two consecutive positive even integers, the product of which is 528.

3. Find two consecutive positive integers, the sum of whose squares is 113.

4. Find two consecutive positive integers, the sum of whose squares is 85.

5–12 ◆ *See Examples 1, 2.*

5. Find the dimensions of a rectangular garden plot if it has an area of 600 square feet and can be surrounded with 100 feet of fence.

6. A fence 240 yards long is used to enclose a rectangular area of 3200 square yards. Find the dimensions of the rectangle.

7. Enclosing a certain rectangular kennel having an area of 1350 square meters requires 150 meters of fence. What are the dimensions of the kennel?

8. The perimeter of a rectangular region is 150 meters. Its area is 1250 square meters. Find the dimensions of the rectangle.

9. The diagonal distance across a rectangular playground is 29 feet. The playground is 1 foot longer than it is wide. Find the dimensions of the playground.

10. The hypotenuse of a right triangle is 17. The sum of its legs is 23. Find the length of each leg.

11. The length of a certain rectangle is 2 feet more than twice its width. Find the dimensions of the rectangle if its area is 60 square feet.

12. The length of a certain rectangle is 2 inches more than 3 times its width. If its area is 33 square inches, find the dimensions of the rectangle.

13–14 ◆ *See Examples 3, 4.*

13. The Rescue Service wishes to send supplies by projectile to a mountaintop 3200 feet above their location. The projectile is fired with an initial upward velocity of 480 feet per second. When do the supplies land (from above) on the mountaintop?

14. A plane flying at an altitude of 1000 feet over the ocean ejects a canister downward with a velocity of 120 feet per second. How long does it take the canister to reach the ocean?

15. The concentration of a pollutant in a river decreases steadily as samples are tested further downstream from the polluting source. The concentration of the pollutant x kilometers from the source is $(10{,}000 - 3x - x^2/20)$ parts per million. How far must one be from the source to find water containing a pollutant concentration of only

    a. 365 parts per million?

    b. 9.8 parts per million?

DEFINITIONS

If there is a constant $k \neq 0$ such that one of the following relationships holds, then k is called a **constant of proportionality** and the terminology of proportion and variation is used as indicated.

1. $y = kx$	y **varies directly** as x, or y is **directly proportional** to x.
2. $y = \dfrac{k}{x}$	y **varies inversely** as x, or y is **inversely proportional** to x.
3. $y = kuv$	y **varies jointly** as u and v, or y is *directly proportional* to u and v.
4. $y = \dfrac{k}{uv}$	y is *inversely proportional* to u and v.

We can expand on these notions of proportionality. For instance, each of the following examples can be written in the given form for some constant $k \neq 0$.

EXAMPLES OF PROPORTIONALITY

1. y is directly proportional to x^4	**1.** $y = kx^4$
2. y is inversely proportional to x^2	**2.** $y = \dfrac{k}{x^2}$
3. y is directly proportional to u^2 and inversely proportional to v^3	**3.** $y = k\dfrac{u^2}{v^3}$
4. y is directly proportional to x^3 and w^2 and inversely proportional to $\sqrt[3]{u}$ and $v^{3/2}$	**4.** $y = \dfrac{x^3w^2}{\sqrt[3]{u}\,v^{3/2}}$

When attempting to develop a mathematical model for a physical phenomenon, the investigator often has some idea regarding direct and inverse variation among variables. The task then is to determine the constant of proportionality k. This can be done by obtaining corresponding values of the variables and solving the proportionality equations for k. The procedure is illustrated in the following examples.

EXAMPLE 1 Let r vary directly as s^2 and assume $r = 50$ when $s = 5$.

a. Write r in terms of s. **b.** Find r when $s = 12$.

Solution **a.** We must find some constant $k \neq 0$ for which

$$r = ks^2$$

When $s = 5$, we have $r = 50$:

$$50 = k \cdot 5^2 = 25k$$
$$k = 2$$

Then $$r = ks^2 = 2s^2.$$

b. When $s = 12$, we have

$$\begin{aligned} r &= 2s^2 = 2 \cdot 12^2 = 2 \cdot 144 \\ &= 288 \end{aligned}$$

◆

EXAMPLE 2 The volume V of a sphere is directly proportional to the cube of its radius r. If the volume is 36π when the radius is 3, write the volume in terms of the radius.

Solution We must find some constant $k \neq 0$ such that

$$V = kr^3$$

When $r = 3$, then $V = 36\pi$:

$$36\pi = k \cdot 3^3 = 27k$$

$$k = \frac{36\pi}{27} = \frac{4\pi}{3}$$

Thus,
$$V = \frac{4\pi}{3} r^3.$$

◆

EXAMPLE 3 If y varies directly as x^2 and inversely as z, and if $y = 50$ when $x = 5$ and $z = 1$, write y in terms of x and z.

Solution We have

$$y = k\frac{x^2}{z}$$

for some constant $k \neq 0$. The given conditions tell us that

$$50 = k\frac{5^2}{1} = 25k$$

That is,
$$k = 2$$

so that
$$y = \frac{2x^2}{z}$$

◆

EXAMPLE 4 Campus Pizza Services delivers pizzas to dormitory rooms at a price proportional to the square of the radius of the pizza. If an 8-inch pizza (radius = 4 inches) sells for \$6.00, what would a 12-inch pizza sell for?

Solution Let C denote the cost of pizza. Then

$$C = kr^2$$

where r is the radius of the pizza. Since $C = 6$ when $r = 4$, we have

$$6 = k \cdot 4^2 = 16k$$

$$k = \frac{6}{16} = \frac{3}{8}$$

Thus
$$C = \frac{3}{8} \cdot r^2$$